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FILLING RADIUS AND SHORT CLOSED GEODESICS OF THE 2-SPHERE

by Stéphane Sabourau

Abstract. — We show that the length of the shortest nontrivial curve among the simple closed geodesics of index zero or one and the figure-eight geodesics of null index provides a lower bound on the area and the diameter of the Riemannian 2-spheres.

Résumé (*Rayon de remplissage et courtes géodésiques fermées de la 2-sphère*)
Nous montrons que la longueur de la plus courte courbe non triviale parmi les géodésiques simples fermées d'indice zéro ou un et les géodésiques en huit d'indice nul fournit une minoration sur l'aire et le diamètre des deux-sphères riemanniennes.

1. Introduction

Let M be a closed connected smooth Riemannian manifold of dimension n. The Riemannian metric g on M induces a distance d_g on M.

The map $i : (M, d_g) \hookrightarrow (L^\infty(M), \|.\|)$ defined by $i(x)(.) = \mathrm{dist}_M(x, .)$ is an embedding from the metric space (M, d_g) into the Banach space $L^\infty(M)$ of all bounded functions on M with the sup-norm $\|.\|$. This natural embedding is a (strong) isometry between metric spaces, *i.e.*, it preserves the distances. Note that Riemannian embeddings of closed manifolds into Euclidean spaces are not isometric in this sense. Considering M isometrically embedded in the

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Banach space $L^\infty(M)$, we define $U_\delta(M)$ as the δ-tubular neighborhood of M in $L^\infty(M)$. The homology coefficients will be in $\mathbb{Z}$, if M is orientable, and in $\mathbb{Z}_2$, otherwise.

Definition. — The filling radius of M, denoted $\operatorname{FillRad}(M)$, is the infimum of positive reals δ such that $(i_\delta)_*([M]) = 0 \in H_n(U_\delta(M))$, where $i_\delta : M \hookrightarrow U_\delta(M)$ is the inclusion and $[M] \in H_n(M)$ is the fundamental class of M.

In this paper, we show the following curvature free estimate.

Theorem 1.1. — *Let M be a Riemannian 2-sphere, then*

$$\operatorname{FillRad}(M) \geq \frac{1}{12}\operatorname{scg}(M) \tag{1.1}$$

where $\operatorname{scg}(M)$ *denotes the length of the shortest nontrivial closed geodesic on M.*

This statement admits a stronger version in which occur the Morse index and the number of self-intersection points of closed geodesics. Note, however, that the constant involved is not as good as in the first version.

Main Theorem 1.2. — *Let M be a Riemannian 2-sphere, then*

$$\operatorname{FillRad}(M) \geq \frac{1}{20}\overline{L}(M) \tag{1.2}$$

where $\overline{L}(M)$ is the length of the shortest nontrivial curve among the simple closed geodesics of index zero or one and the figure-eight geodesics of null index.

For metrics all of whose geodesics are non-degenerate (bumpy metrics) the same inequality holds if we replace $\overline{L}(M)$ by $L(M)$, where $L(M)$ represents the length of the shortest nontrivial curve among the simple closed geodesics of index 1 and the figure-eight geodesics of null index.

Some examples illustrating the different cases of the Main Theorem are presented in this paper (see Remark 4.10).

Before going further, let us review some known results to which these inequalities are related.

These two theorems extend to the simply connected case some filling radius estimates related to the 1-dimensional systole.

The 1-dimensional systole of a non-simply connected closed Riemannian manifold (M, g) is defined as the infimum of the lengths of noncontractible closed curves. This lower bound, denoted $\operatorname{sys}_1(M, g)$, is attained by the length of a closed geodesic.

In [17] (see also [7], [18] and [19]), M. Gromov showed that every essential manifold of dimension n satisfies the isosystolic inequality

$$\operatorname{Vol}(M, g) \geq C_n \operatorname{sys}_1(M, g)^n \tag{1.3}$$

where C_n is a positive constant depending only on n.

In the above statement, whose converse was established in [3], a closed manifold M is said to be essential if there is a map f from M to a $K(\pi,1)$ space such that $f_*([M]) \neq 0$. In particular, $T^n, \mathbb{R}P^n$ and all closed aspherical manifolds are essential.

Isosystolic inequalities on surfaces were previously established in [33], [1], [8], [9, p. 43] and [21]. An inequality similar to (1.3) also holds for the stable systole under suitable topological conditions (see [19], [17], [22], [5], [4] and [24]).

Examples of non-essential manifolds with "long" systole and "small" volume can easily be constructed. The product metric on $S^1 \times S^2$ where the length of S^1 is long and the area of S^2 is small provides such an example. However, they may still have a short contractible closed geodesic whose length is bounded from above in terms of the volume.

For Riemannian 2-spheres, C. Croke showed in [12] that

$$\mathrm{Area}(M) \geq \frac{1}{(31)^2}\,\mathrm{scg}(M)^2, \quad \mathrm{Diam}(M) \geq \frac{1}{9}\,\mathrm{scg}(M).$$

It is unknown whether or not the length of the shortest nontrivial closed geodesic provides a lower bound on the volume of any non-essential manifold of dimension greater than two. Under some curvature assumptions, upper bounds on the length of the shortest nontrivial closed geodesic exist (see [38], [35] and [30] for general results).

The proof of M. Gromov's isosystolic inequality (1.3) rests on the two following filling radius inequalities.

THEOREM 1.3 (M. Gromov). — *Let M be a complete Riemannian n-manifold, then*

$$\mathrm{FillRad}(M) \geq \frac{1}{6}\,\mathrm{sys}_1(M) \ \textit{if } M \textit{ is essential,} \tag{1.4}$$

$$\mathrm{FillRad}(M) \leq c_n\,\mathrm{Vol}(M)^{1/n} \ \textit{for some } c_n > 0. \tag{1.5}$$

In particular, the first inequality holds for all the closed surfaces except the sphere. The second inequality, more difficult to establish (though in the case of the sphere S^2 it may be obtained in a more elementary way), takes the form $\mathrm{FillRad}(S^2) \leq \mathrm{Area}(S^2)^{\frac{1}{2}}$ for the 2-sphere S^2 (see [17, p. 128]).

Thus, the inequalities (1.1) and (1.2) lead to the following corollary which improves C. Croke's result providing an alternative proof.

COROLLARY 1.4. — *Let M be a Riemannian 2-sphere, then*

$$\mathrm{Area}(M) \geq \frac{1}{(12)^2}\,\mathrm{scg}(M)^2, \tag{1.6}$$

$$\mathrm{Area}(M) \geq \frac{1}{(20)^2}\,\overline{L}(M)^2. \tag{1.7}$$

For bumpy metrics, we can replace $\overline{L}(M)$ by $L(M)$ in the above inequalities. Note that the length of a simple closed geodesic around the waist of an hourglass figure does not provide a "good" lower bound on the area as it can be made arbitrarily small while the area remains constant. These closed geodesics, which still are simple and have a null index after slight perturbations of the metric into a bumpy one, can actually be ignored to give a better bound on the area.

Note that inequality (1.6) is not optimal (C. Croke conjectures that the extremal sphere is composed of two copies of flat equilateral triangles glued together along their boundaries) and that sharp isosystolic inequalities are known only for the 2-torus, the projective plane and the Klein bottle (see [7], [33], [6] and [36]).

The proof of the Main Theorem rests on a minimax principle derived from Morse Theory on the space of 1-cycles $\mathcal{Z}_1(M,\mathbb{Z})$ on M. This principle, based on F. Almgren's isomorphism $\pi_1(\mathcal{Z}_1(S^2,\mathbb{Z}),\{0\}) \simeq H_2(S^2,\mathbb{Z}) \simeq \mathbb{Z}$ (see [2] and Theorem 2.4 for a more general version), has been established by F. Almgren and J. Pitts using geometric measure theory and has been used by E. Calabi and J. Cao in [10]. The use of the space of 1-cycles rather than the ordinary free loop space allows us to cut and paste closed curves using several component loops. This minimax principle proceeds as follows.

Let us consider the one-parameter families $(z_t)_{0\leq t\leq 1}$ of 1-cycles on M which satisfy the following conditions:

(C1) z_t starts and ends at null-currents,

(C2) z_t induces a nontrivial class $[z]$ in $\pi_1(\mathcal{Z}_1(M,\mathbb{Z}),\{0\})$.

We define the minimax value

$$L_1(M) := \inf_{[z]\neq 0} \sup_{0\leq t\leq 1} \operatorname{mass}(z_t).$$

For bumpy metrics, we introduce other constructions as follows.

The previous global minimax principle extends to the nontrivial groups $\pi_1(\mathcal{Z}_1^{\leq\kappa_1}(M),\mathcal{Z}_1^{\leq\kappa_0}(M))$, where $0\leq\kappa_0<\kappa_1$ and

$$\mathcal{Z}_1^{\leq\kappa}(M) = \bigl\{z\in\mathcal{Z}_1(M,\mathbb{Z}) \mid \operatorname{mass}(z)\leq\kappa\bigr\}.$$

We refer to Section 4.1 for further details. The lowest positive minimax value of these local minimax processes is noted $L_1'(M)$. We show that $L_1'(M)$ agrees with the mass $L_1''(M)$ of the shortest 1-cycle of index 1. Here, the index of a 1-cycle of mass κ is defined by

$$\operatorname{ind}_{\mathcal{Z}_1}(z) = \min\bigl\{i\in\mathbb{N} \mid \pi_i(\mathcal{Z}_1^{<\kappa}(M)\cup\{z\},\mathcal{Z}_1^{<\kappa}(M)) \text{ is nontrivial}\bigr\}.$$

Further, we show that $\operatorname{scg}(M)\leq L_1'(M)=L_1''(M)\leq L_1(M)$.

We also introduce a new curve-shortening process which permits us to prove the following

THEOREM 1.5. — *Let M be a bumpy Riemannian 2-sphere, then*

$$\mathrm{FillRad}(M) \geq \frac{1}{20} L_1''(M),$$

where $L_1''(M)$ is the length of the shortest 1-cycle of index 1.

We show then that the shortest 1-cycle of index 1 for bumpy metrics is either a simple closed geodesic of index 1 or a figure-eight geodesic of null index. This immediately leads to the Main Theorem.

Contrary to $L_1''(M)$, the invariant $L_1(M)$ provides no universal lower bound on the filling radius of the 2-sphere. More precisely, we have

THEOREM 1.6. — *There exists a sequence g_n of Riemannian metrics on S^2 which satisfies*

$$\lim_{n\to\infty} \frac{\mathrm{FillRad}(S^2, g_n)}{L_1(S^2, g_n)} = 0.$$

Using techniques involved in the proof of Theorem 1.1, we also prove

THEOREM 1.7. — *Let M be a Riemannian 2-sphere of diameter* $\mathrm{Diam}(M)$, *then*

$$\mathrm{scg}(M) \leq 4\,\mathrm{Diam}(M).$$

For other simply connected manifolds, it is still unknown whether or not a similar inequality holds. Note that C. Croke already showed in [12] that $\mathrm{scg}(M) \leq 9\,\mathrm{Diam}(M)$ for 2-spheres. This inequality was then improved with the constant 5 by M. Maeda in [26].

Theorem 1.7 may also be derived from Theorem 1.1 and the sharp general filling inequality $\mathrm{FillRad}(M) \leq \frac{1}{3}\,\mathrm{Diam}(M)$ established by M. Katz in [23]. However, we present its short proof because it illustrates in a simple way some techniques used in this paper.

After having written the final version of this paper, the author learned that A. Nabutovsky and R. Rotman have independently established similar results. Specifically, on 2-spheres, they have obtained in [29] the same improvement for the diameter lower bound as us (*cf.* Theorem 1.7) and a better one for the area lower bound ($\frac{1}{64}$ instead of $\frac{1}{144}$ in (1.6)). They have also obtained in [28] a lower bound on the filling radius of any closed Riemannian manifold in terms of the mass of the shortest stationary 1-cycle.

We refer the reader to the recent survey [13] and the references therein for an account of further curvature-free geometric inequalities.

In Section 2, we study a minimax principle on the space of 1-cycles which yields a nontrivial closed geodesic on the 2-sphere. Then, we introduce a new curve-shortening process. In Section 3, we illustrate the general use of this minimax principle and show that the length of the shortest closed geodesic on the 2-sphere provides a lower bound on the filling radius. Section 4 is devoted to