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ON SQUARE FUNCTIONS ASSOCIATED TO SECTORIAL OPERATORS

by Christian Le Merdy

Dedicated to Alan McIntosh on the occasion of his 60th birthday

Abstract. — We give new results on square functions
$$\|x\|_F = \Big\| \Big(\int_0^\infty \big|F(tA)x\big|^2 \frac{\mathrm{d}t}{t} \Big)^{1/2} \Big\|_p$$
associated to a sectorial operator A on L^p for $1 < p < \infty$. Under the assumption that A is actually R-sectorial, we prove equivalences of the form $K^{-1}\|x\|_G \leq \|x\|_F \leq K\|x\|_G$ for suitable functions F, G. We also show that A has a bounded H^∞ functional calculus with respect to $\|\,.\,\|_F$. Then we apply our results to the study of conditions under which we have an estimate $\|(\int_0^\infty |C\mathrm{e}^{-tA}(x)|^2 \mathrm{d}t)^{1/2}\|_q \leq M\|x\|_p$, when $-A$ generates a bounded semigroup e^{-tA} on L^p and $C\colon D(A) \to L^q$ is a linear mapping.

Résumé (*Sur les fonctions carrées associées aux opérateurs sectoriels*)
Nous obtenons de nouveaux résultats sur les fonctions carrées
$$\|x\|_F = \Big\| \Big(\int_0^\infty \big|F(tA)x\big|^2 \frac{\mathrm{d}t}{t} \Big)^{1/2} \Big\|_p$$
associées à un opérateur sectoriel A sur L^p pour $1 < p < \infty$. Quand A est en fait R-sectoriel, on montre des équivalences de la forme $K^{-1}\|x\|_G \leq \|x\|_F \leq K\|x\|_G$ pour des fonctions F, G appropriées. On démontre également que A possède un calcul fonctionnel H^∞ borné par rapport à $\|\,.\,\|_F$. Puis nous appliquons nos résultats à l'étude de conditions impliquant une inégalité du type $\|(\int_0^\infty |C\mathrm{e}^{-tA}(x)|^2 \mathrm{d}t)^{1/2}\|_q \leq M\|x\|_p$, où $-A$ engendre un semigroupe borné e^{-tA} sur L^p et $C\colon D(A) \to L^q$ est une application linéaire.

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Christian Le Merdy, Département de Mathématiques, Université de Franche-Comté, 25030 Besançon Cedex (France) • *E-mail :* lemerdy@math.univ-fcomte.fr

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1. Introduction

The main objects of this paper will be bounded analytic semigroups and sectorial operators on L^p-spaces, their H^∞ functional calculus, and their associated square functions. This beautiful and powerful subject grew out of McIntosh's seminal paper [18] and subsequent important works by McIntosh-Yagi [19] and Cowling-Doust-McIntosh-Yagi [6].

We first briefly recall a few classical notions which are the starting point of the whole theory. Given a Banach space X, we will denote by $B(X)$ the Banach algebra of all bounded operators on X. For any $\omega \in (0,\pi)$, we let

$$\Sigma_\omega = \big\{ z \in \mathbb{C}^* \, ; \, |\mathrm{Arg}(z)| < \omega \big\}$$

be the open sector of angle 2ω around the half-line $(0,\infty)$. Let A be a possibly unbounded operator A on X and assume that A is closed and densely defined. For any z in the resolvent set of A we let $R(z,A) = (z-A)^{-1}$ denote the corresponding resolvent operator. Let $\sigma(A)$ denote the spectrum of A. Then by definition, A is *sectorial of type* ω if the following three conditions are fulfilled:

(S1) $\sigma(A) \subset \overline{\Sigma}_\omega$.

(S2) For any $\theta \in (\omega,\pi)$ there is a constant $K_\theta > 0$ such that

$$\big\| zR(z,A) \big\| \le K_\theta, \quad z \in \overline{\Sigma}_\theta^{\,c}.$$

(S3) A has a dense range.

Very often, (S3) is unnecessary and omitted in the definition of sectoriality. However we include it here to avoid tedious technical discussions. Note the well-known fact that A is one-to-one if it satisfies (S1), (S2) and (S3) above.

Given any $\theta \in (0,\pi)$, we let $H^\infty(\Sigma_\theta)$ be the algebra of all bounded analytic functions $f : \Sigma_\theta \to \mathbb{C}$ and we let $H_0^\infty(\Sigma_\theta)$ be the subalgebra of all $f \in H^\infty(\Sigma_\theta)$ for which there exist two positive numbers $s, c > 0$ such that

$$(1.1) \qquad \big| f(z) \big| \le c \frac{|z|^s}{(1+|z|)^{2s}}, \quad z \in \Sigma_\theta.$$

Now given a sectorial operator A of type $\omega \in (0,\pi)$ on a Banach space X, a number $\theta \in (\omega,\pi)$, and a function $f \in H_0^\infty(\Sigma_\theta)$, one may define an operator $f(A) \in B(X)$ as follows. We let $\gamma \in (\omega,\theta)$ be an intermediate angle and consider the oriented contour Γ_γ defined by

$$\Gamma_\gamma(t) = \begin{cases} -t\mathrm{e}^{i\gamma} & t \in \mathbb{R}_-, \\ t\mathrm{e}^{-i\gamma} & t \in \mathbb{R}_+. \end{cases}$$

Then we let

$$(1.2) \qquad f(A) = \frac{1}{2\pi i} \int_{\Gamma_\gamma} f(z) R(z,A) \, \mathrm{d}z.$$

It follows from Cauchy's Theorem that the definition of $f(A)$ does not depend on the choice of γ and it can be shown that the mapping $f \mapsto f(A)$ is an algebra homomorphism from $H_0^\infty(\Sigma_\theta)$ into $B(X)$. The next step in H^∞ functional calculus consists in the definition of a possibly unbounded operator $f(A)$ associated to any $f \in H^\infty(\Sigma_\theta)$. Since we shall not use this construction here, we omit it and refer the reader to [18], [19] and [6] for details. We merely recall that by definition, A admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus if $f(A)$ is bounded for any $f \in H^\infty(\Sigma_\theta)$. In that case, the mapping $f \mapsto f(A)$ is a bounded homomorphism from $H^\infty(\Sigma_\theta)$ into $B(X)$, provided that $H^\infty(\Sigma_\theta)$ is equipped with the norm

$$\|f\|_{\infty,\theta} = \sup\bigl\{|f(z)| \,;\, z \in \Sigma_\theta\bigr\}.$$

We shall be mainly concerned by square functions associated to sectorial operators in the case when X is an L^p-space. For any $\omega \in (0, \pi)$, we introduce

$$H_0^\infty(\Sigma_{\omega+}) = \bigcup_{\theta>\omega} H_0^\infty(\Sigma_\theta).$$

Assume first that $X = H$ is a Hilbert space. Given a sectorial operator A of type ω on H and $F \in H_0^\infty(\Sigma_{\omega+})$, we consider

$$\|x\|_F = \Bigl(\int_0^\infty \|F(tA)x\|^2 \frac{\mathrm{d}t}{t}\Bigr)^{1/2}, \quad x \in H,$$

which may be either finite or infinite. These square function norms were introduced in [18] where it is shown that for any $\theta > \omega$ and any non zero $F \in H_0^\infty(\Sigma_{\omega+})$, A has a bounded $H^\infty(\Sigma_\theta)$ functional calculus if and only if $\|.\|_F$ is equivalent to the original norm of H. In [19, Theorem 5], McIntosh-Yagi established the following two remarkable properties. First these square function norms are pairwise equivalent, that is, for any two non zero functions F and G in $H_0^\infty(\Sigma_{\omega+})$ there exists a constant $K > 0$ such that $K^{-1}\|x\|_G \leq \|x\|_F \leq K\|x\|_G$ for any $x \in H$. Second, A always has a bounded H^∞ functional calculus with respect to $\| \ \|_F$. More precisely, for any $\theta > \omega$ and for any $F \in H_0^\infty(\Sigma_\theta)$, there is a constant $K > 0$ such that $\|f(A)x\|_F \leq K\|f\|_{\infty,\theta}\|x\|_F$ for any $f \in H^\infty(\Sigma_\theta)$ and any $x \in H$. Further properties and applications of square functions $\|.\|_F$ were investigated in [3], to which we refer the interested reader.

We now turn to L^p-spaces. Let $1 \leq p < \infty$ be a number, let Ω be an arbitrary measure space, and consider the Banach space $X = L^p(\Omega)$. Given a sectorial operator A of type ω on $L^p(\Omega)$ and $F \in H_0^\infty(\Sigma_{\omega+})$, we let

$$\|x\|_F = \Bigl\|\Bigl(\int_0^\infty \bigl|F(tA)x\bigr|^2 \frac{\mathrm{d}t}{t}\Bigr)^{1/2}\Bigr\|_{L^p(\Omega)}, \quad x \in L^p(\Omega).$$

Again $\|x\|_F$ may be either finite or infinite. These square function norms were introduced in [6] and play a key role in the study of bounded H^∞ functional calculus on L^p-spaces (see Corollary 2.3 below). The latter definition obviously extends the previous one that we recover when $p = 2$. However it is unknown whether the results from [19] reviewed above extend to the case when $p \neq 2$. In particular it is unknown whether square function norms are pairwise equivalent on L^p-spaces. In a recent work [2], Auscher-Duong-McIntosh succeded in proving such an equivalence in the case when $-A$ generates a bounded analytic semigroup acting on $L^2(\Omega)$ with suitable upper bounds on its heat kernels. We shall prove that the results from [19, Theorem 5] actually extend to all operators which are not only sectorial but R-sectorial. This notion which arose from some recent work of Weis [22] will be explained at the beginning of the next section.

THEOREM 1.1. — *Let A be an R-sectorial operator of R-type $\omega \in (0,\pi)$ on a space $L^p(\Omega)$, with $1 \leq p < \infty$. Let $\theta \in (\omega,\pi)$ and let F and G be two non zero functions belonging to $H_0^\infty(\Sigma_\theta)$.*

1) *There exists a constant $K > 0$ such that for any $f \in H^\infty(\Sigma_\theta)$ and any $x \in L^p(\Omega)$, we have*

$$\Big\| \Big(\int_0^\infty \big| f(A)F(tA)x \big|^2 \frac{\mathrm{d}t}{t} \Big)^{1/2} \Big\|_{L^p(\Omega)} \leq K \|f\|_{\infty,\theta} \Big\| \Big(\int_0^\infty \big| G(tA)x \big|^2 \frac{\mathrm{d}t}{t} \Big)^{1/2} \Big\|_{L^p(\Omega)}. \tag{1.3}$$

2) *There exists a constant $K > 0$ such that*

$$K^{-1}\|x\|_G \leq \|x\|_F \leq K\|x\|_G, \quad x \in L^p(\Omega).$$

This result will be proved in Section 2 below, where we also include some relevant comments. Then Section 3 is devoted to an application of Theorem 1.1 to the study of R-admissibility. This new concept is a natural extension of the classical notion of admissibility considered *e.g.* in [24], [23], [25], [8] or [16]. Given a bounded analytic semigroup $T_t = \mathrm{e}^{-tA}$ on $L^p(\Omega)$ and a linear mapping C from the domain of A into some $L^q(\Sigma)$, we will study conditions under which we have an estimate of the form

$$\Big\| \Big(\int_0^\infty \big| CT_t(x) \big|^2 \mathrm{d}t \Big)^{1/2} \Big\|_{L^q(\Sigma)} \leq M \|x\|_{L^p(\Omega)}.$$

In particular we will show that such an estimate holds if A has a bounded $H^\infty(\Sigma_\theta)$ functional calculus for some $\theta < \frac{1}{2}\pi$ and the set $\big\{(-s)^{1/2}CR(s,A)\ ;\ s \in \mathbb{R},\ s < 0\big\}$ is R-bounded. This extends a result of ours ([16]) corresponding to the case when $p = 2$.

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2. Equivalence of square function norms

The main purpose of this section is the proof of Theorem 1.1. We first recall the key concepts of R-boundedness (see [4]) and R-sectoriality (see [22], [21], [14]). Consider a Rademacher sequence $(\varepsilon_k)_{k\geq 1}$ on a probability space $(\Omega_0, \mathbb{P})$. That is, the ε_k's are pairwise independent random variables on Ω_0 and $\mathbb{P}(\varepsilon_k = 1) = \mathbb{P}(\varepsilon_k = -1) = \frac{1}{2}$ for any $k \geq 1$. Then for any finite family $x_1, \ldots, x_n$ in a Banach space X, we let

$$\Big\|\sum_{k=1}^{n} \varepsilon_k x_k\Big\|_{\mathrm{Rad}(X)} = \int_{\Omega_0} \Big\|\sum_{k=1}^{n} \varepsilon_k(s) x_k\Big\|_X \mathrm{d}\mathbb{P}(s).$$

Let X, Y be two Banach spaces and let $B(X,Y)$ denote the space of all bounded operators from X into Y. By definition, a set $\mathcal{T} \subset B(X,Y)$ is R-bounded if there is a constant $C \geq 0$ such that for any finite families $T_1, \ldots, T_n$ in $\mathcal{T}$, and $x_1, \ldots, x_n$ in X, we have

$$\Big\|\sum_{k=1}^{n} \varepsilon_k T_k(x_k)\Big\|_{\mathrm{Rad}(Y)} \leq C \Big\|\sum_{k=1}^{n} \varepsilon_k x_k\Big\|_{\mathrm{Rad}(X)}.$$

In that case, the smallest possible C is called the *R-boundedness constant* of $\mathcal{T}$ and is denoted by $R(\mathcal{T})$. If A is a sectorial operator on X and $\omega \in (0, \pi)$ is a number, we say that A is *R-sectorial of R-type* ω if for any $\theta \in (\omega, \pi)$, the set $\{zR(z,A)\ ;\ z \in \overline{\Sigma}_\theta^c\} \subset B(X)$ is R-bounded.

To describe the range of applications of our result, we first recall that if X is a Hilbert space, then any bounded subset of $B(X)$ is R-bounded, hence any sectorial operator of type ω on X is actually R-sectorial of R-type ω. Thus Theorem 1.1 comprises [19, Theorem 5] that we recover when $p = 2$. Note that our proof reduces to that of [19] in this case. If X is not isomorphic to a Hilbert space, then there exist bounded subsets of $B(X)$ which are not R-bounded (see *e.g.* [1, Proposition 1.13]). The notion of R-sectoriality on non Hilbertian Banach spaces is closely related to maximal L^p-regularity. Namely, it was proved in [13] and [22] that if A is a sectorial operator of type $< \frac{1}{2}\pi$ on a Banach space X with maximal L^p-regularity, then A is R-sectorial of R-type $< \frac{1}{2}\pi$. Thus the counterexamples to maximal L^p-regularity obtained by Kalton-Lancien [13] show that when $p \neq 2$, there exist sectorial operators on L^p-spaces which are not R-sectorial. Conversely, it was proved in [22] that if X is a UMD Banach space, and A is R-sectorial of R-type $< \frac{1}{2}\pi$ on X, then A has maximal L^p-regularity. Thus for $1 < p < \infty$ and $\omega < \frac{1}{2}\pi$, Theorem 1.1 exactly applies when the operator A has maximal L^p-regularity. In particular it applies to the operators considered in [2].