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EXPONENTIALS FORM A BASIS OF DISCRETE HOLOMORPHIC FUNCTIONS ON A COMPACT

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ABSTRACT. — We show that discrete exponentials form a basis of discrete holomorphic functions on a finite critical map. On a combinatorially convex set, the discrete polynomials form a basis as well.

RÉSUMÉ (*Les exponentielles forment une base des fonctions holomorphes discrètes sur un compact*)

Nous montrons que les exponentielles forment une base des fonctions holomorphes discrètes sur une carte critique compacte. Sur un convexe, les polynômes discrets forment également une base.

1. Introduction

The notion of discrete Riemann surfaces has been defined in [13]. A good basis for the associated space of holomorphic functions was still missing. This article discuss an interesting one in the simply connected critical case.

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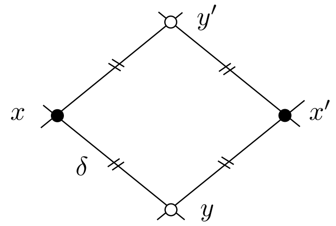

FIGURE 1. The discrete Cauchy-Riemann equation takes place on each rhombus.

We are interested in a cellular decomposition $\diamondsuit$ of the complex plane or a simply connected portion U of it, by *rhombi* (equilateral quadrilaterals, or lozenges). In other words, our cellular complex is made of quadrilaterals (a *quad-graph*) and we have a map from the set of vertices $\diamondsuit_0$ to the complex plane $Z : \diamondsuit_0 \to \mathbb{C}$ such that for each oriented face $(x, y, x', y') \in \diamondsuit_2$, its image is a positively oriented rhombus $(Z(x), Z(y), Z(x'), Z(y'))$ of side length $\delta > 0$. It defines a straightforward Cauchy-Riemann equation for a function $f \in C^0(\diamondsuit)$ of the vertices, and similarly for 1-forms:

$$\frac{f(y') - f(y)}{Z(y') - Z(y)} = \frac{f(x') - f(x)}{Z(x') - Z(x)}. \tag{1.1}$$

We call such a data a *critical map* of U. The relevance of this kind of maps in the context of discrete holomorphy was first pointed out and put to use by Duffin [8]. The rhombi can be split in four, yielding a finer critical map $\diamondsuit'$, Z' with $\delta' = \frac{1}{2}\delta$. In [13], [12] we proved that a converging sequence of discrete holomorphic functions on a refining sequence of critical maps converges to a continuous holomorphic function and any holomorphic function on U can be approximated by a converging sequence of discrete holomorphic functions. The proof was based on discrete polynomials and series. In the present article we are going to show that the vector space spanned by discrete polynomials is the same as the one spanned by discrete exponentials and the main result is the following

THEOREM 1. — *On a combinatorially convex finite critical map, the discrete exponentials form a basis of discrete holomorphic functions.*

On a non combinatorially convex map we define some special exponentials which supplement this basis.

The article is organized as follows. After recalling some basic features of discrete Riemann surfaces at criticality in Section 2, we define discrete exponentials in Section 3 and show some of their basic properties, related to polynomials and series. We give in particular a formula for the expression of a generic exponential in a basis of exponentials. In Section 4, we introduce the notion of convexity, related to a geometrical construction called train-tracks, and we prove the main result. Finally we study the general case in Section 5 where we define special exponentials and show they form a basis. The appendix lists some other interesting properties of the discrete exponentials which are not needed in the proof.

We note that it is possible to use the wonderful machinery defined in [10] to prove that discrete exponentials form a basis of discrete holomorphic functions on a critical compact. Indeed, Richard Kenyon gives an expression of the discrete Green's function (the discrete logarithm) as an integral over a loop in the space of discrete exponentials:

$$G(O,x) = -\frac{1}{8\,\pi^2\, i}\oint_C \exp(:\!\lambda\!: x)\,\frac{\log \frac{1}{2}\delta\lambda}{\lambda}\,\mathrm{d}\lambda \tag{1.2}$$

where the integration contour C contains all the points in $P_\diamondsuit$ (the possible poles of $\exp(:\!\lambda\!: x)$) but avoids the half line through $-x$. It is real (negative) on half of the vertices and imaginary on the others. Because of the logarithm, this imaginary part is multivalued, it has a (discrete) logarithmic singularity at the origin: the Laplacian of $G(O,\bullet)$ is 1 there, and null elsewhere. On a compact, considering points on the boundary as origins, these functions can be single valued and can be formed into a basis of discrete holomorphic functions. They clearly belong in the space of discrete exponentials.

The approach we will present here is much more pedestrian and simplistic.

In a forthcoming paper in collaboration with A. Bobenko, B. Springborn and Y. Suris, the theory will be generalized to a much broader setup, based on a quadratic notion of discrete holomorphicity: cross-ratio preserving maps. Given a base function F defined on a quad-graph, a function G is *cross-ratio holomorphic* [3], [2], [4] if, for every quadrilateral $(x,y,x',y')\in\diamondsuit_2$,

$$\frac{G(x)-G(y)}{G(y)-G(x')}\cdot\frac{G(x')-G(y')}{G(y')-G(x)} = \frac{F(x)-F(y)}{F(y)-F(x')}\cdot\frac{F(x')-F(y')}{F(y')-F(x)}\cdot \tag{1.3}$$

The circle packings of a given combinatorics and intersection angles form a very interesting example of such cross-ratio preserving maps.

Infinitesimal deformations of a function F with given cross-ratios, that preserve these cross-ratios, are parametrized by the vector space $\Omega(F)$ of discrete linear holomorphic functions g defined by the fact that their ratios along the diagonals of the quadrilaterals are the same as the base function: g is *linear holomorphic* at F if

$$\frac{g(x)-g(x')}{g(y)-g(y')} = \frac{F(x)-F(x')}{F(y)-F(y')}\cdot \tag{1.4}$$

An analytic vector field of linear holomorphic functions can be integrated into cross-ratio preserving maps. For example, the vector field given by the Green function is integrated into the 1-parameter line Z^γ [1].

An important classical ingredient in the theory of cross-ratio preserving maps is the *Bäcklund transformation* [4], [5], [9]. It is a two complex parameters (u,λ) family of (cross-ratio) holomorphic functions $B^u_\lambda(F)$, that is to say with the same cross-ratios as F. The parameter u is a starting value at a given origin, $B^u_\lambda(F)(O)=u$. This transformation verifies

$$B^{F(O)}_{-\lambda}\big(B^u_\lambda(F)\big)=F \tag{1.5}$$

for any (u,λ), and the identity transformation corresponds to $(u,\lambda)=(F(O),0)$, $B^{F(O)}_0(F)=F$. It is an anlytic transformation in all the parameters therefore its derivative is a linear map between the tangent spaces,

$$\mathrm{d}B^u_\lambda(F):\Omega(F)\longrightarrow\Omega\big(B^u_\lambda(F)\big). \tag{1.6}$$

It is not injective and I define the discrete exponential at F as being the direction of this 1-dimensional kernel. It can be characterized as a derivative with respect to the initial value at the origin:

$$\exp_u(:\lambda:F):=\frac{\partial}{\partial v}B^v_{-\lambda}\big(B^u_\lambda(F)\big)_{|v=F(O)}\in\ker\big(\mathrm{d}B^u_\lambda(F)\big) \tag{1.7}$$

because $B^u_\lambda(B^v_{-\lambda}(G))=G$ for all λ, G and v. We use them to derive Kenyon's formula in a more general setup and cover the subject of an infinite critical map with finitely many slopes, extending the results of the present article.

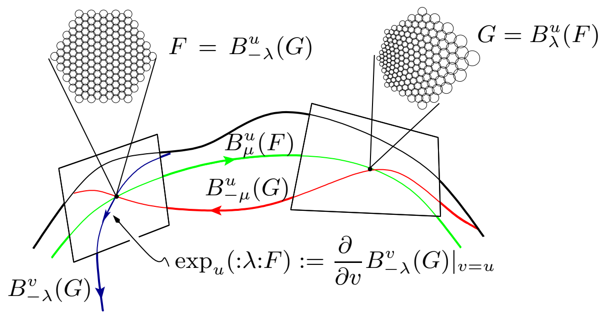

FIGURE 2. The discrete exponential $\exp_u(:\lambda:F)$ is the kernel of the linear transformation $\mathrm{d}B^u_\lambda(F)$ (here $u=F(O)$).

2. Integration and Derivation at criticality

2.1. Integration. — Given an isometric local map $Z:U\cap\diamondsuit\to\mathbb{C}$, where the image of the quadrilaterals are lozenges in $\mathbb{C}$, any holomorphic function $f\in\Omega(\diamondsuit)$ gives rise to an holomorphic 1-form $f\,\mathrm{d}Z$ defined by the formula,

$$\int_{(x,y)}f\,\mathrm{d}Z:=\frac{f(x)+f(y)}{2}\big(Z(y)-Z(x)\big), \tag{2.1}$$

where $(x,y)\in\diamondsuit_1$ is an edge of a lozenge. It fulfills the Cauchy-Riemann equation for forms which is, in the same conditions as Eq. (1.1):

$$\frac{1}{Z(y')-Z(y)}\Big(\int_{(y,x)}+\int_{(x,y')}+\int_{(y,x')}+\int_{(x',y')}\Big)f\,\mathrm{d}Z \\ =\frac{1}{Z(x')-Z(x)}\quad\Big(\int_{(x,y)}+\int_{(y,x')}+\int_{(x,y')}+\int_{(y',x')}\Big)f\,\mathrm{d}Z. \tag{2.2}$$

Once an origin O is chosen, it provides a way to integrate a function $\operatorname{Int}(f)(z) := \int_O^z f\,\mathrm{d}Z$. We proved in [12] that the integrals of converging discrete holomorphic functions $(f^k)_k$ on a refining sequence $(\diamondsuit^k)_k$ of critical maps of a compact converge to the integral of the limit. If the original limit was of order $f(z) = f^k(z) + O(\delta_k^2)$, it stays this way for the integrals, $\int_O^z f(u)\mathrm{d}u = \int_O^z f^k\,\mathrm{d}Z + O(\delta_k^2)$, where the left hand side is the usual continuous integral and the right hand side the discrete ones.

Following Duffin [7], [8], we define by inductive integration the discrete analogues of the integer power monomials z^k, that we denote $Z^{:k:}$:

$$Z^{:0:} := 1, \tag{2.3}$$

$$Z^{:k:} := k\int_O Z^{:k-1:}\mathrm{d}Z. \tag{2.4}$$

The discrete polynomials of degree less than three agree point-wise with their continuous counterpart, $Z^{:2:}(x) = Z(x)^2$ so that by repeated integration, the discrete polynomials in a refining sequence of a compact converge to the continuous ones and the limit is of order $O(\delta^2)$. We will see (Eq. (3.6)) that a closed formula can be obtained for these monomials.

2.2. Derivation. — The combinatorial surface being simply connected and the graph $\diamondsuit$ having only quadrilateral faces, it is bi-colorable. Let Γ and Γ^* the two sets of vertices and ε be the *biconstant* $\varepsilon(\Gamma) = +1$, $\varepsilon(\Gamma^*) = -1$. For a holomorphic function f, the equality $f\,\mathrm{d}Z \equiv 0$ is equivalent to $f = \lambda\,\varepsilon$ for some $\lambda\in\mathbb{C}$.

Following Duffin [7], [8], we introduce the

Definition 2.1. — For a holomorphic function f, define on a flat simply connected map U the holomorphic functions $f^\dagger$, the *dual* of f, and f', the derivative of f, by the formulae

$$f^\dagger(z) := \varepsilon(z)\,\bar f(z), \tag{2.5}$$

where $\bar f$ denotes the complex conjugate, $\varepsilon = \pm 1$ is the biconstant, and

$$f'(z) := \frac{4}{\delta^2}\Big(\int_O^z f^\dagger\,\mathrm{d}Z\Big)^\dagger + \lambda\,\varepsilon, \tag{2.6}$$

defined up to ε.