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DUAL BLOBS AND PLANCHEREL FORMULAS

by Ju-Lee Kim

Abstract. — Let k be a p-adic field. Let G be the group of k-rational points of a connected reductive group G defined over k, and let $\mathfrak{g}$ be its Lie algebra. Under certain hypotheses on G and k, we *quantify* the tempered dual $\widehat{G}$ of G via the Plancherel formula on $\mathfrak{g}$, using some character expansions. This involves matching spectral decomposition factors of the Plancherel formulas on $\mathfrak{g}$ and G. As a consequence, we prove that any tempered representation contains a good minimal K-type; we extend this result to irreducible admissible representations.

Résumé (*Blobs duaux et formule de Plancherel*). — Soient k un corps p-adique, G un groupe réductif connexe défini sur k, G son groupe de points k-rationnels et $\mathfrak{g}$ l'algèbre de Lie de G. Sous certaines hypothèses, nous *quantifions* le dual tempéré $\widehat{G}$ de G par la formule de Plancherel sur $\mathfrak{g}$, en utilisant des développements en caractères. Pour cela, il faut en particulier mettre en correspondance les facteurs de la décomposition spectrale de la formule de Plancherel sur $\mathfrak{g}$ et sur G. Comme conséquence, nous démontrons que toute représentation tempérée contient un bon K-type minimal ; nous étendons aussi ce résultat aux représentations admissibles irréductibles.

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Introduction

Let k be a p-adic field, G a connected reductive group defined over k, and G the group of k-rational points of G. Let $\widetilde{G}$ be the set of the equivalence classes of irreducible admissible representations of G. To study $\widetilde{G}$, representations of compact open subgroups of G have been useful. In particular, Moy and Prasad studied open compact subgroups coming from the theory of the Bruhat-Tits building $\mathcal{B}(G)$ of G. They introduced (unrefined) minimal K-types of the form $\mathfrak{s} := (G_{x,\varrho}, \chi)$, where $G_{x,\varrho}$ is an open compact subgroup of G associated to $(x, \varrho) \in \mathcal{B}(G) \times \mathbb{R}_{\geq 0}$, and χ is an irreducible representation of $G_{x,\varrho}$ of a certain type. If we introduce *weak associativity*, an equivalence relation of minimal K-types (see Definition 2.2.1), we can partition $\widetilde{G}$ as follows

$$\widetilde{G} = \mathop{\overset{\circ}{\bigcup}}_{\mathfrak{s} \in \mathfrak{S}_{\mathsf{K}}} \widetilde{G}_{\mathfrak{s}},$$

where $\widetilde{G}_{\mathfrak{s}}$ is the set of $(\pi, V_\pi) \in \widetilde{G}$ containing a minimal K-type weakly associated to $\mathfrak{s}$, and $\mathfrak{S}_{\mathsf{K}}$ is the set of equivalence classes of weakly associated minimal K-types.

Let $\widehat{G}$ be the tempered dual of G. Recall that $\widehat{G}$ is the support of the Plancherel measures in the unitary dual of G. Let $\widehat{\mathfrak{g}}$ be the unitary dual of $\mathfrak{g}$. Then $\widehat{\mathfrak{g}}$ is also a tempered dual of $\mathfrak{g}$. In this paper, we "quantify" the *Plancherel integral* over each $\widehat{G}_{\mathfrak{s}} := \widehat{G} \cap \widetilde{G}_{\mathfrak{s}}$ in terms of the *Plancherel integral* over an appropriate G-domain in the Lie algebra $\mathfrak{g}$ of G, when the residue characteristic of k is large. Based on the Kirillov theory of compact p-adic groups [10], such a quantification first appeared in the work of Howe [11]. To start with, we observe that from Plancherel formulas on $\mathfrak{g}$ and G, we have

$$\text{(Pl)} \qquad \int_{\widehat{\mathfrak{g}} \simeq \mathfrak{g}} \widehat{f}(X) \mathrm{d}X = f(0) = \int_{\widehat{G}} \Theta_\pi(f \circ \log) \mathrm{d}\pi$$

for any $f \in C_c^\infty(\widehat{\mathfrak{g}}) \simeq C_c^\infty(\mathfrak{g})$ supported in a small neighborhood of 0 (here, we identify $\mathfrak{g}$ and $\widehat{\mathfrak{g}}$ via Pontrjagin duality). Now, we can refine this equality: we find an equality between spectral decomposition factors of each side (see 1.4.5 and 3.2.6) of (Pl), where spectral components of the right hand side are parameterized by certain minimal K-types, and those of the left hand side by some related G-domains in $\widehat{\mathfrak{g}}$.

To be more precise, from now on, we fix a *good* minimal K-type $\mathfrak{s} = (G_{x,\varrho}, \chi)$. Note that the depth of $\mathfrak{s}$ is ϱ. A minimal K-type is *good* when its *dual blob* (the dual coset realizing χ) is good (see [3] or 1.2.2, 3.2.1). Let $\mathcal{S}$ be the dual blob of $\mathfrak{s}$, and let $\widetilde{G}_{\mathcal{S}} \subset \widetilde{G}_{\mathfrak{s}}$ be the set of $(\pi, V_\pi) \in \widetilde{G}_{\mathfrak{s}}$ containing good K-types weakly associated to $\mathcal{S}$ (in this paper, we introduce three weak associativities: between minimal K-types (2.2.1), between good cosets (1.4.1), between good K-types and good cosets (3.2.3)). Then we match the Plancherel integral over $\widehat{G}_{\mathcal{S}} := \widehat{G} \cap \widetilde{G}_{\mathcal{S}}$

with the Fourier transform of some distribution supported on a G-domain $\mathfrak{g}_{\mathcal{S}}$ in $\mathfrak{g}$ coming from $\mathcal{S}$. Roughly speaking, $\mathfrak{g}_{\mathcal{S}}$ is the G-orbit of good dual blobs weakly associated to $\mathfrak{s}$ (see 1.4.3). In particular, for two good minimal K-types $\mathfrak{s}'$ and $\mathfrak{s}''$ with dual blobs $\mathcal{S}'$ and $\mathcal{S}''$, if $\widetilde{G}_{\mathfrak{s}'} \cap \widetilde{G}_{\mathfrak{s}''} = \varnothing$, then $\mathfrak{g}_{\mathcal{S}'} \cap \mathfrak{g}_{\mathcal{S}''} = \varnothing$. Moreover, $\mathfrak{g} = \mathring{\cup} \mathfrak{g}_{\mathcal{S}'}$ (see 1.4.5) where $\mathcal{S}'$ runs over weakly associated classes of good cosets. We remark that the depth of any $(\pi, V_\pi) \in \widetilde{G}_{\mathcal{S}}$ is the same as the depth ϱ of $\mathfrak{s}$, and the depth of $\mathfrak{g}_{\mathcal{S}}$ (see Definition 1.2.3) is $-\varrho$. Let $\mathfrak{g}_\varrho := \bigcup_{x \in \mathcal{B}(G)} \mathfrak{g}_{x,\varrho}$. When $\varrho > 0$, we prove that for any $f \in C_c^\infty(\mathfrak{g}_\varrho)$,

$$\int_{\mathfrak{g}_{\mathcal{S}}} \widehat{f}(X) \mathrm{d}X = \int_{\widehat{G}_{\mathcal{S}}} \Theta_\pi (f \circ \log) \mathrm{d}\pi. \tag{1}$$

Summing over $\mathcal{S}$ in weakly associated classes of good cosets, we see

$$\begin{aligned} \int_{\mathfrak{g}} \widehat{f}(X) \mathrm{d}X &= \sum_{\mathcal{S}} \int_{\mathfrak{g}_{\mathcal{S}}} \widehat{f}(X) \mathrm{d}X \\ &= \sum_{\mathcal{S}} \int_{\widehat{G}_{\mathcal{S}}} \Theta_\pi (f \circ \log) \mathrm{d}\pi = \int_{\widehat{G}} \Theta_\pi (f \circ \log) \mathrm{d}\pi, \end{aligned} \tag{2}$$

which leads to the proof that every tempered representation contains a good minimal K-type (see Theorem 4.5.1) when the residue characteristic is large. Under the same hypothesis, we also show that any irreducible admissible representation contains a good minimal K-type. This fact has been already proved (see [13, 2.4.10]) using more tools from the theory of buildings. Although we still use such tools quite a bit, this work is just an initial step to approach the problem on the exhaustion of the types constructed in [12]. In this sense, the purpose of this paper is rather a piecewise *quantification*, *i.e.*, a matching of spectral decomposition factors as in (1) (see Theorem 3.3.1). However, one can hope that this analytic approach might be more fruitful for generalization.

To prove the equality in (1), we regard both sides of (1) as distributions on $C_c^\infty(\mathfrak{g}_\varrho)$. Here, the domain where (1) holds is restricted to $\mathfrak{g}_\varrho$, however, this is large enough to single out $\widetilde{G}_{\mathfrak{s}}$ from $\widetilde{G}$ in the following sense: there are $f \in C_c^\infty(\mathfrak{g}_\varrho)$ such that $\Theta_\pi(f \circ \log) \neq 0$ implies $\pi \in \widetilde{G}_{\mathfrak{s}}$. On the other hand, $C_c^\infty(\mathfrak{g}_{\varrho^+})$ (here, $\mathfrak{g}_{\varrho^+} = \bigcup_{s > \varrho} \mathfrak{g}_s$) can not isolate better than the set of depth ϱ representations which strictly contains $\widetilde{G}_{\mathfrak{s}}$. In the case of depth zero representations, it is not possible to distinguish a single class of K-types by dual cosets. Hence replacing $\mathfrak{g}_{\mathcal{S}}$ and $\widetilde{G}_{\mathcal{S}}$ by a G-domain $\mathfrak{g}_0$ and the set of all depth zero representations $\widetilde{G}_0$, we prove an analogous equality on $C_c^\infty(\mathfrak{g}_{0^+})$. However, this is good enough to prove that any tempered representation contains a good minimal K-type.

We approach this problem via various character expansions and homogeneity results. When $\varrho = 0$, by the work of Waldspurger and DeBacker (see [18], [5]), we know that the Harish-Chandra-Howe local character expansion is valid on the set $\mathfrak{g}_{0^+}$ of topologically nilpotent elements in $\mathfrak{g}$. We show that the distributions in (1) are in the span of Fourier transforms of nilpotent orbital integrals when restricted to $\mathfrak{g}_{0^+}$. Then we match two distributions using Gelfand-Graev functions as test functions as in [4].

If $\varrho > 0$, the Harish-Chandra-Howe expansions are not enough, because the set $\mathfrak{g}_{\varrho^+}$ where they hold is not big enough. Hence we use Γ-asymptotic expansions of representations in $\widetilde{G}_{\mathcal{S}}$. In [13], F. Murnaghan and the author proved that such expansions are valid on the G-domain $\mathfrak{g}_\varrho$. More precisely, assume that π contains a *good* type coming from a good element Γ of depth $-\varrho$ (see [3] or 1.2.1–1.2.2 for definition). Denote the set of G-orbits whose closure contains Γ by $\mathcal{O}(\Gamma)$. Then we can express the character distribution Θ_π as a linear combination of Fourier transforms of orbital integrals $\mu_{\mathcal{O}}$ with $\mathcal{O} \in \mathcal{O}(\Gamma)$. That is, if π is an irreducible admissible representation of G containing $(G_{x,\varrho}, \chi)$, and if χ is realized by a good element Γ, we prove that there are $c_{\mathcal{O}}(\pi) \in \mathbb{C}$ indexed by $\mathcal{O}(\Gamma)$ such that

$$\Theta_\pi(\exp X) = \sum_{\mathcal{O} \in \mathcal{O}(\Gamma)} c_{\mathcal{O}}(\pi) \widehat{\mu_{\mathcal{O}}}(X),$$

and this expansion is valid on $\mathfrak{g}_\varrho \cap \mathfrak{g}_{\mathrm{reg}}$. Then we show that the distributions in (1) are in the span of Fourier transforms of $\mu_{\mathcal{O}}$ with $\mathcal{O} \in \mathcal{O}(\Gamma)$ when restricted to $\mathfrak{g}_\varrho$, and we match two distributions using some test functions found in [13].

In the first section, we recall some basic definitions related to this work, and state the hypotheses that we use at various places of this paper. We also define the weak associativity of good cosets, which induces a partition of $\mathfrak{g}$ accordingly (§1.4). Then we show that this partition induces a spectral decomposition as in (2) (see Lemma 1.4.5). In Section 2, we define the weak associativity of minimal K-types, which induces a partition of $\widetilde{G}$. In Section 3, we relate the partitions on $\mathfrak{g}$ and $\widetilde{G}$ found in the first two sections. Section 4 is basically devoted to proving the equality in (1) (see Theorem 3.3.1). As an application, in the end of Section 4, we prove that any tempered representation contains a good minimal K-type; we also discuss the extension of this result to irreducible admissible representations. As a corollary, we also get a new spectral decomposition of the delta distribution on G where each decomposition factor is parameterized by the G-orbit of a good dual blob.

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Notation and Conventions. — Let k be a p-adic field (a finite extension of $\mathbb{Q}_p$) with residue field $\mathbb{F}_{p^n}$. Let $\nu = \nu_k$ be the valuation on k such that

$\nu(k^\times) = \mathbb{Z}$. Let $\bar{k}$ be an algebraic closure of k. For an extension field E of k, let ν_E be the valuation on E extending ν. We will just write ν for ν_E. Let $\mathcal{O}_E$ be the ring of integers of E with prime ideal $\mathfrak{p}_E$. Let Λ be a fixed additive character of k such that $\Lambda_{|\mathcal{O}_k} \neq 1$ and $\Lambda_{|\mathfrak{p}_k} = 1$.

Let G be a connected reductive group defined over k, and $\mathsf{G}(E)$ the group of E-rational points of G. We denote by G the group of k-rational points of G. Denote the Lie algebras of G and $\mathsf{G}(E)$ by $\boldsymbol{\mathfrak{g}}$ and $\boldsymbol{\mathfrak{g}}(E)$, respectively. Denote by $\boldsymbol{\mathfrak{g}}^*$ and $\boldsymbol{\mathfrak{g}}^*(E)$ the linear duals of $\boldsymbol{\mathfrak{g}}$ and $\boldsymbol{\mathfrak{g}}(E)$ respectively. We write $\mathfrak{g}$ and $\mathfrak{g}^*$ for the vector space of k-rational points of $\boldsymbol{\mathfrak{g}}$ and $\boldsymbol{\mathfrak{g}}^*$ respectively. In general, we use bold characters $\mathsf{H}, \mathsf{M}, \mathsf{N}$, *etc.* to denote algebraic groups defined over k, corresponding Roman characters H, M and N to denote the groups of k-points, and $\mathfrak{h}, \mathfrak{m}$ and $\mathfrak{n}$ to denote the Lie algebras of H, M and N.

Denote the set of regular elements in $\mathfrak{g}$ by $\mathfrak{g}_{\mathrm{reg}}$. Let $\mathcal{N}$ be the set of nilpotent elements in $\mathfrak{g}$. There are different notions of nilpotency. However, since we assume that $\mathrm{char}(k) = 0$, those notions are all the same. We refer to [14], [6] for more discussion of this.

If X is a topological space with a Borel measure $\mathrm{d}x$ and if Y is a subset of X, $\mathrm{vol}_X(Y)$ denotes the volume of Y with respect to $\mathrm{d}x$.

For any subset S in $\mathfrak{g}$, we denote by $[S]$ the characteristic function of S, and by $-S$ the set $\{-s \mid s \in S\}$. For $g \in G$, ${}^gZ$ denotes gZg^{-1} and for $H \subset G$, ${}^HS$ denotes $\{\,{}^gZ \mid Z \in S,\ g \in H\}$.

Let $\widetilde{G}$ be the set of equivalence classes of irreducible admissible representations of G. Let $\widehat{G}$ be the subset of $\widetilde{G}$ which consists of equivalence classes of tempered representations of G.

1. Good cosets and $\mathfrak{g}$

1.1. Moy-Prasad filtrations. — For a finite extension E of k, let $\mathcal{B}(\mathsf{G}, E)$ denote the extended Bruhat-Tits building of G over E. For a maximal torus T in G, if it splits over E, let $\mathcal{A}(\mathsf{T}, E)$ be the corresponding apartment over E. It is known that if E' is a tamely ramified Galois extension of E, $\mathcal{B}(\mathsf{G}, E)$ can be embedded into $\mathcal{B}(\mathsf{G}, E')$ and its image is equal to the set of the Galois fixed points in $\mathcal{B}(\mathsf{G}, E')$ (see [17, 5.11] or [16]). Moreover, we have

$$\mathcal{A}(\mathsf{T}, E) = \mathcal{A}(\mathsf{T}, E') \cap \mathcal{B}(\mathsf{G}, E).$$

We let $\mathcal{A}(\mathsf{T}, k) := \mathcal{A}(\mathsf{T}, E) \cap \mathcal{B}(\mathsf{G}, k)$.

Regarding G as a group defined over E, Moy and Prasad associate $\boldsymbol{\mathfrak{g}}(E)_{x,r}$ and $\mathsf{G}(E)_{x,|r|}$ to $(x, r) \in \mathcal{B}(\mathsf{G}, E) \times \mathbb{R}$ with respect to the valuation normalized as follows [14]: let E^u be the maximal unramified extension of E, and L the minimal extension of E^u over which G splits. Then the valuation used by Moy and Prasad maps $L^\times$ onto $\mathbb{Z}$.