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ON MEROMORPHIC FUNCTIONS DEFINED BY A DIFFERENTIAL SYSTEM OF ORDER 1

by Tristan Torrelli

Abstract. — Given a germ h of holomorphic function on $(\mathbb{C}^n, 0)$, we study the condition: "the ideal $\mathrm{Ann}_{\mathcal{D}} 1/h$ is generated by operators of order 1". We obtain here full characterizations in the particular cases of Koszul-free germs and unreduced germs of plane curves. Moreover, we prove that this condition holds for a special type of hyperplane arrangements. These results allow us to link this condition to the comparison of de Rham complexes associated with h.

Résumé (*Sur les germes de fonctions méromorphes définis par un système différentiel d'ordre* 1)

Étant donné un germe de fonction holomorphe h défini au voisinage de l'origine de $\mathbb{C}^n$, nous étudions la condition : « l'idéal $\mathrm{Ann}_{\mathcal{D}} 1/h$ est engendré par des opérateurs d'ordre 1 ». Nous obtenons ici des caractérisations complètes dans le cas des germes Koszul-libres et dans celui des germes de courbes planes non réduits. De plus, nous montrons que cette condition est vérifiée pour un type particulier d'arrangements d'hyperplans. Ces résultats nous permettent de relier cette condition à la comparaison de complexes de de Rham associés à h.

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1. Introduction

Let $h \in \mathcal{O} = \mathbb{C}\{x_1, \dots, x_n\}$ be a nonzero germ of holomorphic function such that $h(0) = 0$. We denote by $\mathcal{O}[1/h]$ the ring $\mathcal{O}$ localized by the powers of h. Let $\mathcal{D} = \mathcal{O}\langle \partial/\partial x_1, \dots, \partial/\partial x_n \rangle$ be the ring of linear differential operators with holomorphic coefficients and $F_\bullet\mathcal{D}$ its filtration by order. As usual, we identify $\mathrm{gr}^F\mathcal{D}$ with the polynomial ring $\mathcal{O}[\xi] = \mathcal{O}[\xi_1, \dots, \xi_n]$.

Given $a/h^\ell \in \mathcal{O}[1/h]$ nonzero, we consider the following condition:

> *The left ideal* $\mathrm{Ann}_\mathcal{D}\, a/h^\ell \subset \mathcal{D}$ *of operators annihilating* a/h^ℓ *is generated by operators of order* 1.

This condition appears when studying the elements of the holonomic $\mathcal{D}$-modules $\mathcal{O}[1/h]$ and $\mathcal{O}[1/h]/\mathcal{O}$ (see [18]). Moreover, it is directly linked to the so-called "Logarithmic Comparison Theorem" (see below). The aim of this work is to explicit this condition. First we remark the following fact.

Proposition 1.1. — *Let $a, h \in \mathcal{O}$ be germs of holomorphic functions without common factor. If the ideal $\mathrm{Ann}_\mathcal{D}\, a/h$ is generated by operators of order 1, then a is a unit.*

So, without loss of generality, we will suppose that $a = 1$. When h defines a hypersurface with isolated singularity, we have obtained in [18] the following characterization.

Theorem 1.2. — *Let $h \in \mathcal{O}$ be a germ of a holomorphic function defining an isolated singularity. Let $\ell \in \mathbb{N}^*$ be a nonnegative integer. Then the ideal $\mathrm{Ann}_\mathcal{D}\, 1/h^\ell$ is generated by operators of order 1 if and only if the following conditions are verified:*

(a) *the germ h is weighted-homogeneous,*

(b) *the smallest integral root of the Bernstein polynomial of h is strictly greater than $-\ell - 1$.*

We recall that a nonzero germ h is *weighted-homogeneous* of weight $d \in \mathbb{Q}^+$ for a system $\alpha \in (\mathbb{Q}^{*+})^n$ if there exists a system of coordinates in which h is a linear combination of monomials $x_1^{\gamma_1} \cdots x_n^{\gamma_n}$ with $\sum_{i=1}^n \alpha_i \gamma_i = d$. Moreover, the condition (b) means that $1/h^\ell$ generates the $\mathcal{D}$-module $\mathcal{O}[1/h]$ (see [11, Prop. 6.2] and [2, Prop. 6.1.18, 6.3.15 & 6.3.16]; for the definition of the Bernstein polynomial, see the beginning of part 2).

What does remain true without any assumption on h ? First of all, the condition (b) is always necessary.

Proposition 1.3. — *Let $h \in \mathcal{O}$ be a nonzero germ of holomorphic function with $h(0) = 0$. Let $\ell \in \mathbb{N}^*$ be a nonnegative integer such that the ideal $\mathrm{Ann}_\mathcal{D}\, 1/h^\ell$ is generated by operators of order 1. Then the smallest integral root of the Bernstein polynomial of h is strictly greater than $-\ell - 1$.*

On the other hand, h is not always weighted-homogeneous (Example 1.5). So, let us denote the condition:

(a′) h belongs to the ideal of its partial derivatives.

In other words, there exists a vector field $v \in \mathcal{D}$ such that $v(h) = h$, and we will say that h is *Euler-homogeneous.* In the case of hypersurfaces with isolated singularities, K. Saito has proved that these two conditions coincide (see [13]). We conjecture the following fact.

Conjecture 1.4. — *If there exists a nonnegative integer $\ell \in \mathbb{N}^*$ such that $\operatorname{Ann}_{\mathcal{D}} 1/h^\ell$ is generated by some operators of order 1, then h is Euler-homogeneous.*

Reciprocally, conditions (a′) and (b) are not always enough to have $\operatorname{Ann}_{\mathcal{D}} 1/h^\ell$ generated by operators of order 1 (see Example 1.9). Nevertheless, they are sufficient when the ideal $\operatorname{Ann}_{\mathcal{D}} h^s$ is generated by operators of order 1 (this is true in the case of isolated singularities (see [12, p. 117], or [23, Thm 2.19])). Indeed, if h is Euler-homogeneous, then we have a decomposition:

$$\operatorname{Ann}_{\mathcal{D}[s]} h^s = \mathcal{D}[s](s - v) + \mathcal{D}[s] \operatorname{Ann}_{\mathcal{D}} h^s;$$

moreover, with the condition (b), $\operatorname{Ann}_{\mathcal{D}} 1/h^\ell$ is obtained by fixing $s = -\ell$ in a system of generators of $\operatorname{Ann}_{\mathcal{D}[s]} h^s$ (see [18, Prop. 3.1]). Finally, the fact that $\operatorname{Ann}_{\mathcal{D}} 1/h^\ell$ is generated by operators of order 1 does not imply that so is $\operatorname{Ann}_{\mathcal{D}} h^s$.

Example 1.5 (see [3], [4], [6]). — Let $h = x_1x_2(x_1 + x_2)(x_1 + x_2x_3)$. It is an Euler-homogeneous polynomial which is not weighted-homogeneous. Indeed, if there exists a change of coordinates $\varphi = (\varphi_1, \varphi_2, \varphi_3)$ – with $\varphi(0) = 0$ – such that $h \circ \varphi$ is a weighted-homogeneous polynomial for $\alpha \in (\mathbb{Q}^{*+})^3$, then its factors are weighted-homogeneous too. Thus the polynomials φ_1, φ_2 and $\varphi_2\varphi_3$ must have the same weight, and this is absurd.

The ideal $\operatorname{Ann}_{\mathcal{D}} 1/h$ is generated by the operators:

$$\begin{aligned}
S_1 &= (x_1 + x_2x_3)\frac{\partial}{\partial x_3} + x_2,\\
S_2 &= x_2(x_1 + x_2)\frac{\partial}{\partial x_2} - x_1(x_3 - 1)\frac{\partial}{\partial x_3} + x_1 + 3x_2,\\
S_3 &= x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + 4.
\end{aligned}$$

The $\mathcal{O}$-module $\operatorname{Ann}_{\mathcal{D}} h^s \cap F_1\mathcal{D}$ is generated by:

$$Q_1 = 4S_1 - x_2S_3, \quad Q_2 = 4S_2 - (x_1 + 3x_2)S_3$$

and it defines an ideal $I \subset \mathcal{D}$ which does not coincide with $\operatorname{Ann}_{\mathcal{D}} h^s$. Indeed, one can verify that the following operator:

$$\begin{aligned} P = 2x_2^2\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2} - 2x_2^2\frac{\partial^2}{\partial x_2} - 2(x_1+3x_2)x_3\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_3} \\ +2(x_1-2x_2+5x_2x_3)\frac{\partial}{\partial x_2}\frac{\partial}{\partial x_3} + 8(1-x_3)x_3\frac{\partial^2}{\partial x_3} \\ -x_1\frac{\partial}{\partial x_1} - x_2\frac{\partial}{\partial x_2} - 4(2x_3+1)\frac{\partial}{\partial x_3} \end{aligned}$$

annihilates h^s. But P does not belong to I because the ideal $\operatorname{gr}^F I$ is generated by the principal symbols $\sigma(Q_1), \sigma(Q_2)$, and in particular $\operatorname{gr}^F I \subset (x_1, x_2)\mathcal{O}[\xi]$ even if $\sigma(P) \notin (x_1, x_2)\mathcal{O}[\xi]$.

In the two following parts, we try to extend to other situations the characterization given by Theorem 1.2. We begin with the case of plane curves.

THEOREM 1.6. — *Let $h \in \mathbb{C}\{x_1, x_2\}$ be nonzero with $h(0) = 0$, and let $\ell \in \mathbb{N}^*$ be a nonnegative integer.*

(i) *The ideal $\operatorname{Ann}_{\mathcal{D}} 1/h^\ell$ is generated by operators of order 1 if and only if h is weighted-homogeneous.*

(ii) *Let $N \in \mathbb{N}^*$ be a nonnegative integer greater than or equal to 2. Let $\tilde{b}(s) \in \mathbb{C}[s]$ be the reduced Bernstein polynomial of h. Then the ideal $\operatorname{Ann}_{\mathcal{D}} 1/(h+x_3^N)^\ell$ is generated by operators of order 1 if and only if the following conditions are verified:*

(a) *the germ h is weighted-homogeneous,*

(b) *$\ell \geq 2$, or $\ell = 1$ and -2 is not a root of a polynomial $\tilde{b}(s+i/N)$, for $1 \leq i \leq N-1$.*

If h is reduced, it is a very particular case of Theorem 1.2 (for another proof of (i), see [6]). We use that the Euler-homogeneous germs of plane curves are weighted-homogeneous (Proposition 3.4), which comes from K. Saito ([13]).

Another part is devoted to a variant of Theorem 1.2, where the assumption on h is replaced by a condition on the graded ideal of $\operatorname{Ann}_{\mathcal{D}} 1/h^\ell$.

THEOREM 1.7. — *Let $h \in \mathcal{O}$ be a nonzero germ such that $h(0) = 0$, and $\ell \in \mathbb{N}^*$. Suppose that the $\mathcal{O}$-module $\operatorname{Ann}_{\mathcal{D}} 1/h^\ell \cap F_1\mathcal{D}$ is generated by operators $Q_1, \ldots, Q_w$ such that: $\operatorname{gr}^F \mathcal{D}(Q_1, \ldots, Q_w) = \big(\sigma(Q_1), \ldots, \sigma(Q_w)\big)\operatorname{gr}^F\mathcal{D}$. Then the ideal $\operatorname{Ann}_{\mathcal{D}} 1/h^\ell$ is generated by a system of operators of order 1 if and only if the following conditions are verified:*

(a) *the germ h belongs to the ideal of its partial derivatives,*

(b) *the smallest integral root of the Bernstein polynomial of h is strictly greater than $-\ell-1$,*

(c) *the ideal $\operatorname{Ann}_{\mathcal{D}} h^s$ is generated by operators of order 1.*

Moreover, $\operatorname{Ann}_{\mathcal{D}} h^s$ *is also generated by* $Q_j(1)Q_i - Q_i(1)Q_j$, $1 \leq i \leq w$, $i \neq j$, *where* j *is such that* $Q_j(1)$ *is a unit.*

It is not easy to find a family of germs which verify this assumption. Except for the case of weighted-homogeneous isolated singularities (see [19, Prop. 4.3]), one can prove that it is also verified for a particular type of free germs – in the sense of K. Saito [14]: the so-called *Koszul-free* germs.

Recall that a reduced germ $h \in \mathcal{O}$ is *free* if the $\mathcal{O}$-module $\operatorname{Der}(\log h) \subset \mathcal{D}$ of vector fields v such that $v(h) \in h\mathcal{O}$ is free (its rank is also equal to n). The germ h is said to be *Koszul-free* if there exists a basis $\{\delta_1, \ldots, \delta_n\}$ of $\operatorname{Der}(\log h)$ such that the sequence of principal symbols $(\sigma(\delta_1), \ldots, \sigma(\delta_n))$ is $\operatorname{gr}^F\mathcal{D}$-regular (see [3]). For example, germs of reduced plane curves and locally weighted-homogeneous free germs are Koszul-free (see [14, Cor. 1.7] and [4]).

Corollary 1.8. — *Let* $h \in \mathcal{O}$ *be a Koszul-free germ. Then the ideal* $\operatorname{Ann}_{\mathcal{D}} 1/h$ *is generated by operators of order* 1 *if and only if the following conditions are verified:*

(a) *the germ* h *is Euler-homogeneous,*

(b) -1 *is the only integral root of the Bernstein polynomial of* h*,*

(c) *the ideal* $\operatorname{Ann}_{\mathcal{D}} h^s$ *is generated by operators of order* 1.

Suppose furthermore that h *is Euler-homogeneous. Let* $\{\delta_1, \ldots, \delta_n\}$ *be a basis of* $\operatorname{Der}(\log h)$ *such that* $\delta_1(h) = h$ *and* $\delta_i(h) = 0$, $2 \leq i \leq n$. *Then condition* (c) *is equivalent to:*

(c′) *the sequence* $(h, \sigma(\delta_2), \ldots, \sigma(\delta_n))$ *is* $\operatorname{gr}^F\mathcal{D}$*-regular.*

The following example shows that condition (c) is neither a consequence of the assumption of Theorem 1.7 on $\operatorname{gr}^F \operatorname{Ann}_{\mathcal{D}} 1/h^\ell$ nor a consequence of conditions (a) and (b) for a Koszul-free germ.

Example 1.9. — Let $\tilde{h} = x_1^5 + x_2^5 + x_1^2 x_2^2$. It is a Koszul-free germ which is not Euler-homogeneous. Let $h = \exp(x_3)\tilde{h}$. Using Saito criterion (see [14]), it is easy to see that the Euler-homogeneous germ h is Koszul-free. Up to a unit, h and $\tilde{h}$ are equal ; so they have the same Bernstein polynomial. In particular, -1 is the only integral root of the Bernstein polynomial of h. So h verifies conditions (a) and (b), but not (c). Indeed, condition "$\operatorname{Ann}_{\mathcal{D}} 1/h$ is generated by operators of order 1" only depends on the hypersurface germ defined by h, and it is not verified by $\tilde{h}$ (see Theorem 1.6).

Let us remark that this characterization can not be extented to the case of free germs (since the germ of Example 1.5 is free).

In the last part, we study the case of a hyperplane arrangement defined by $h = 0$ in $\mathbb{C}^n$. Indeed, A. Leykin has proved the following fact.