Bull. Soc. math. France
133 (1), 2005, p. 121–144

CHARACTERIZATION OF CYCLE DOMAINS VIA KOBAYASHI HYPERBOLICITY

by Gregor Fels & Alan Huckleberry

Abstract. — A real form G of a complex semi-simple Lie group $G^{\mathbb{C}}$ has only finitely many orbits in any given $G^{\mathbb{C}}$-flag manifold $Z = G^{\mathbb{C}}/Q$. The complex geometry of these orbits is of interest, e.g., for the associated representation theory. The open orbits D generally possess only the constant holomorphic functions, and the relevant associated geometric objects are certain positive-dimensional compact complex submanifolds of D which, with very few well-understood exceptions, are parameterized by the Wolf cycle domains $\Omega_W(D)$ in $G^{\mathbb{C}}/K^{\mathbb{C}}$, where K is a maximal compact subgroup of G. Thus, for the various domains D in the various ambient spaces Z, it is possible to compare the cycle spaces $\Omega_W(D)$.

The main result here is that, with the few exceptions mentioned above, for a fixed real form G all of the cycle spaces $\Omega_W(D)$ are the same. They are equal to a universal domain Ω_{AG} which is natural from the the point of view of group actions and which, in essence, can be explicitly computed.

The essential technical result is that if $\widehat{\Omega}$ is a G-invariant Stein domain which contains Ω_{AG} and which is Kobayashi hyperbolic, then $\widehat{\Omega} = \Omega_{AG}$. The equality of the cycle domains follows from the fact that every $\Omega_W(D)$ is itself Stein, is hyperbolic, and contains Ω_{AG}.

Texte reçu le 12 février 2003, révisé le 14 octobre 2003

Gregor Fels, Fakultät für Mathematik, Universität Tübingen, Auf der Morgenstelle 14, 72076 (Germany) • *E-mail :* `gfels@uni-tuebingen.de` • *Url :* `www.mathematik.uni-tuebingen.de`

Alan Huckleberry, Fakultät für Mathematik, Ruhr-Universität Bochum, Universitätsstrasse 150, 44801 Bochum (Germany) • *E-mail :* `ahuck@cplx.ruhr-uni-bochum.de` • *Url :* `www.cplx.ruhr-uni-bochum.de`

2000 Mathematics Subject Classification. — 22E46, 32M05.

Key words and phrases. — Complex geometry, cycles spaces, Lie groups, Schubert varieties.

G.F.: Research supported by a Habilitationsstipendium from the Deutsche Forschungsgemeinschaft.

A.H.: Research partially supported by grants from the Deutsche Forschungsgemeinschaft and the Japanese Society for the Promotion of Science.

BULLETIN DE LA SOCIÉTÉ MATHÉMATIQUE DE FRANCE 0037-9484/2005/121/$ 5.00

Résumé (*Caractérisation de domaines de cycles par l'hyperbolicité au sens de Kobayashi*)
Une forme réelle G d'un groupe de Lie semi-simple $G^{\mathbb{C}}$ n'admet qu'un nombre fini d'orbites dans toute $G^{\mathbb{C}}$-variété de drapeaux $Z = G^{\mathbb{C}}/Q$. La géométrie complexe de ces orbites est intéressante, par exemple pour la théorie de la représentation associée. Les fonctions holomorphes sur les orbites ouvertes D de G sont constantes en général ; les objets géométriques importants liés à ces orbites sont des sous-variétés complexes de D de dimension positives qui, à quelques rares exceptions bien comprises, sont paramétrées par les domaines de cycles de Wolf $\Omega_W(D) \in G^{\mathbb{C}}/K^{\mathbb{C}}$, où K est un sous-groupe maximal compact de G. Alors, pour les domaines D dans les variétés ambiantes Z, il est possible de comparer les domaines de cycles $\Omega_W(D)$.
Le résultat principal de cet article, aux exceptions près mentionnées ci-dessus, est que pour une forme réelle G fixée, les domaines $\Omega_W(D)$ sont les mêmes. Ils sont égaux à un domaine universel Ω_{AG}, qui est canonique du point de vue d'actions de groupe et qui peut être essentiellement calculé.
Le résultat technique important est que tout domaine de Stein hyperbolique au sens de Kobayashi $\widehat{\Omega}$ qui contient Ω_{AG} est égal à Ω_{AG}. L'égalité des domaines de cycles s'ensuit du fait que chaque $\Omega_W(D)$ est lui-même de Stein, hyperbolique et contient Ω_{AG}.

1. Introduction

Let G be a non-compact real semi-simple Lie group which is embedded in its complexification $G^{\mathbb{C}}$ and consider the associated G-action on a $G^{\mathbb{C}}$-flag manifold $Z = G^{\mathbb{C}}/Q$. It is known that G has only finitely many orbits in Z; in particular, there exit open G-orbits D. In each such open orbit every maximal compact subgroup K of G has exactly one orbit C_0 which is a complex submanifold (see [42]).

Let $q := \dim_{\mathbb{C}} C_0$, regard C_0 as a point in the space $\mathcal{C}^q(Z)$ of q-dimensional compact cycles in Z and let

$$\Omega := G^{\mathbb{C}} \cdot C_0$$

be the orbit in $\mathcal{C}^q(Z)$. Define the Wolf cycle space $\Omega_W(D)$ to be the connected component of $\Omega \cap \mathcal{C}^q(D)$ which contains the base cycle C_0.

Since the above mentioned basic paper [42], there has been a great deal of work aimed at describing these cycle spaces. Even in situations where good matrix models are available this is not a simple matter. Using a variety of techniques, exact descriptions of $\Omega_W(D)$ have been given in a number of special situations (see *e.g.* [1] [4], [3], [13], [17], [23], [24], [34], [38], [41], [45]).

In [16] it was conjectured that, except in the holomorphic Hermitian case where $\Omega_W(D)$ is just the associated bounded symmetric space, the cycle spaces can be naturally identified with a certain universal domain Ω_{AG} which only depends on G. This domain, which is precisely defined below, is a certain G-invariant neighborhood of the Riemannian symmetric space $M = G/K$ in its complexification $\Omega = G^{\mathbb{C}}/K^{\mathbb{C}}$.

The inclusion $\Omega_{AG} \subset \Omega_W(D)$ was proved in most cases in [17] by analyzing concrete models and by using a nice general result which reduces this inclusion to special cases.

In [25], using incidence geometry given by Schubert varieties (see also [23] and [22]), it was shown that $\Omega_W(D)$ agrees with the Schubert domain $\Omega_S(D)$ which is defined by removing certain algebraic incidence divisors from Ω.

The Schubert domains in turn contain a universal domain Ω_I which is known to agree with Ω_{AG}. The inclusion $\Omega_{AG} \subset \Omega_I$ was proved by complex analytic methods (see [22]), but now there is an algebraic proof (see [33]) which may be more appropriate, because the situation would apriori seem to be algebraic in nature. The inclusion $\Omega_I \subset \Omega_{AG}$ was shown in [2]. Thus,

$$\Omega_{AG} \subset \Omega_I \subset \Omega_S(D) = \Omega_W(D).$$

In particular $\Omega_{AG} \subset \Omega_W(D)$, has now been proved in complete generality. Therefore, to prove the above mentioned conjecture it is necessary to prove the opposite inclusion $\Omega_W(D) \subset \Omega_{AG}$.

This is a consequence of the following complex geometric characterization of Ω_{AG} which is the main result of the present paper (see Theorem 3.4.5).

Theorem 1.0.1. — *If $\widehat{\Omega}$ is a G-invariant domain which contains Ω_{AG} in Ω and which is in addition Stein and Kobayashi hyperbolic, then $\widehat{\Omega} = \Omega$.*

Obviously $\widehat{\Omega} = \Omega_W(D)$ is G-invariant. It follows directly from the definitions that Schubert domains are Stein (see [25]). Thus $\Omega_W(D) = \Omega_S(D)$ implies that the cycle spaces are Stein, a fact that has been known in the measurable case for some time (see [43]).

Using a slight refinement of the results in [22], we show here that, with the exception of the holomorphic Hermitian case where $\Omega_W(D)$ is just the associated bounded symmetric space, $\Omega_W(D)$ is naturally embedded in Ω as a Kobayashi hyperbolic domain.

Consequently, with this well-understood exception, the above theorem together with the inclusion $\Omega_{AG} \subset \Omega_W(D)$ shows that $\Omega_W(D) = \Omega_{AG}$.

Before going to the main body of our work, let us set the notation.

Let $M = G/K$ be the associated Riemannian symmetric space of non-positive curvature embedded in $M^{\mathbb{C}} = G^{\mathbb{C}}/K^{\mathbb{C}}$ as an orbit of the same base point x_0 as was chosen above in the discussion of cycle spaces.

Denote by θ a Cartan involution on $\mathfrak{g}^{\mathbb{C}}$ which restricts to a Cartan involution on $\mathfrak{g}$ such that $\mathrm{Fix}(\theta_{|\mathfrak{g}}) = \mathfrak{k}$ is the Lie algebra of the given maximal compact subgroup K. The anti-holomorphic involution $\sigma : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ which defines $\mathfrak{g}$ commutes with θ as well as with the holomorphic extension τ of $\theta_{|\mathfrak{g}}$ to $\mathfrak{g}^{\mathbb{C}}$.

Let $\mathfrak{u}$ be the fixed point set of θ in $\mathfrak{g}^{\mathbb{C}}$, U be the associated maximal compact subgroup of $G^{\mathbb{C}}$ and define Σ to be the connected component containing x_0 of $\{x \in U \cdot x_0 : G_x \text{ is compact}\}$. Set $\Omega_{AG} := G \cdot \Sigma$.

To cut down on the size of Σ, one considers a maximal Abelian subalgebra $\mathfrak{a}$ in $\mathfrak{p}$ (where $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the Cartan decomposition of $\mathfrak{g}$) and notes that $G \cdot (\exp(i\mathfrak{a}) \cap \Sigma) \cdot x_0 = \Omega_{AG}$.

In fact there is an explicitly defined neighborhood ω_{AG} of $0 \in \mathfrak{a}$ such that $i\omega_{AG}$ is mapped diffeomorphically onto its images $\exp(i\omega_{AG})$ and $\exp(i\omega_{AG}) \cdot x_0$ and $\Omega_{AG} = G \cdot \exp(i\omega_{AG}) \cdot x_0$.

The set ω_{AG} is defined by the set of roots $\Phi(\mathfrak{a})$ of the adjoint representation of $\mathfrak{a}$ on $\mathfrak{g}$: It is the connected component containing $0 \in \mathfrak{a}$ of the set which is obtained from $\mathfrak{a}$ by removing the root hyperplanes $\{\mu = \frac{1}{2}\pi\}$ for all $\mu \in \Phi$. It is convex and is invariant under the action of the Weyl group $\mathcal{W}(\mathfrak{a})$ of the symmetric space G/K.

Modulo $\mathcal{W}(\mathfrak{a})$, the set $\exp(i\omega_{AG}) \cdot x_0$ is a geometric slice for the G-action on Ω_{AG}. From this root point of view, Σ can be seen to be the set of points which are at most half way from x_0 to the cut-point locus in the compact Riemannian symmetric space U/K (see [11]).

2. Spectral properties of $\mathbf{\Omega_{AG}}$

2.1. Linearization. — The map

$$\eta : G^{\mathbb{C}} \longrightarrow \operatorname{Aut}_{\mathbb{R}}(\mathfrak{g}^{\mathbb{C}}), \quad x \longmapsto \sigma \circ \operatorname{Ad}(x) \circ \tau \circ \operatorname{Ad}(x^{-1}),$$

provides a suitable linearization of the setting at hand. The idea of using this linearization in the context of double coset spaces is due to T. Masuki. Some of the results in this and the following section on the Jordan decomposition can be found in §4 of [32]. In particular, in §3.2 for the sake of completeness we give proofs of his Proposition 3 and Proposition 4. In this section elementary properties of η are summarized.

Let $G^{\mathbb{C}}$ act on $\operatorname{Aut}_{\mathbb{R}}(\mathfrak{g}^{\mathbb{C}})$ by $h \cdot \varphi := \operatorname{Ad}(h) \circ \varphi \circ \operatorname{Ad}(h^{-1})$.

Lemma 2.1.1 (G-equivariance). — *For $h \in G$ it follows that $\eta(h \cdot x) = h \cdot \eta(x)$ for all $x \in G^{\mathbb{C}}$.*

Proof. — By definition $\eta(h \cdot x) = \sigma \operatorname{Ad}(h) \operatorname{Ad}(x) \tau \operatorname{Ad}(x^{-1}) \operatorname{Ad}(h^{-1})$. Since h belongs to G, it follows that σ and $\operatorname{Ad}(h)$ commute and the desired result is immediate. □

The normalizer of $K^{\mathbb{C}}$ in $G^{\mathbb{C}}$ is denoted by $N^{\mathbb{C}} := N_{G^{\mathbb{C}}}(K^{\mathbb{C}})$. It is indeed the complexification of $N := N_U(K)$.

Lemma 2.1.2 ($N^{\mathbb{C}}$-invariance). — *The map η factors through a G-equivariant embedding of $G^{\mathbb{C}}/N^{\mathbb{C}}$:*

$$\eta(x) = \eta(y) \iff y = xg^{-1} \text{ for some } g \in N^{\mathbb{C}}.$$

Proof. — We may write $y = xg^{-1}$ for some $g \in G^{\mathbb{C}}$. Thus it must be shown that $\eta(x) = \eta(xg^{-1})$ if and only if $g \in N^{\mathbb{C}}$. But $\eta(x) = \eta(xg^{-1})$ is equivalent to $\mathrm{Ad}(g)\tau = \tau\,\mathrm{Ad}(g)$, which, in turn, is equivalent to the fact that $\mathrm{Ad}(g)$ stabilizes the complexified Cartan decomposition $\mathfrak{g}^{\mathbb{C}} = (\mathfrak{g}^{\mathbb{C}})^{\tau} \oplus (\mathfrak{g}^{\mathbb{C}})^{-\tau} = \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{p}^{\mathbb{C}}$.

Now, if $\mathrm{Ad}(g)$ stabilizes $\mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{p}^{\mathbb{C}}$, then $\mathrm{Ad}(g)(\mathfrak{k}^{\mathbb{C}}) = \mathfrak{k}^{\mathbb{C}}$, *i.e.*, $g \in N^{\mathbb{C}}$. On the other hand, given any $g \in N^{\mathbb{C}}$, it follows $\mathrm{Ad}(g)(\mathfrak{p}^{\mathbb{C}}) = \mathfrak{p}^{\mathbb{C}}$, because $\mathfrak{p}^{\mathbb{C}}$ is the orthogonal complement of $\mathfrak{k}^{\mathbb{C}}$ with respect to the Killing form of $\mathfrak{g}^{\mathbb{C}}$. □

Note that $N^{\mathbb{C}}/K^{\mathbb{C}}$ is a finite Abelian group (see [15] for a classification). Consequently, up to finite covers, η is an embedding of the basic space $G^{\mathbb{C}}/K^{\mathbb{C}}$.

The involutions σ and τ are regarded as acting on $\mathrm{Aut}_{\mathbb{R}}(\mathfrak{g}^{\mathbb{C}})$ by conjugation. On $\mathrm{Im}(\eta)$ their behavior is particularly simple.

LEMMA 2.1.3 (Action of the basic involutions). — *For all $x \in G^{\mathbb{C}}$ it follows that*

1) $\eta(\tau(x)) = \tau(\eta(x))$,
2) $\sigma(\eta(x)) = \eta(x)^{-1}$,

In particular $\mathrm{Im}(\eta)$ *is both σ- and τ-invariant.*

Proof. — Let $\varphi_* : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ denote the differential of $\varphi : G^{\mathbb{C}} \to G^{\mathbb{C}}$ and $\mathrm{Int}(x) : G^{\mathbb{C}} \to G^{\mathbb{C}}$ be defined by $\mathrm{Int}(x)(z) := xzx^{-1}$. The first statement follows directly from the facts that σ and τ commute and

$$\tau\,\mathrm{Ad}(x)\tau = \big(\tau\,\mathrm{Int}(x)\tau\big)_* = \mathrm{Int}\big(\tau(x)\big)_* = \mathrm{Ad}\big(\tau(x)\big).$$

For the second statement note that $\sigma\eta(x) = \mathrm{Ad}(x)\tau\,\mathrm{Ad}(x^{-1})$, and thus $\eta(x)\sigma\eta(x) = \sigma$. □

We have seen that η is a G-equivariant map which induces a finite equivariant map $\eta : G^{\mathbb{C}}/K^{\mathbb{C}} \to \mathrm{Aut}_{\mathbb{R}}(\mathfrak{g}^{\mathbb{C}})$. We will shortly see that the image $\eta(G^{\mathbb{C}}/K^{\mathbb{C}})$ is also closed in $\mathrm{Aut}_{\mathbb{R}}(\mathfrak{g}^{\mathbb{C}})$. Hence, for a characterization of G-orbits in $G^{\mathbb{C}}/K^{\mathbb{C}}$ and their topological properties we may identify $G^{\mathbb{C}}/K^{\mathbb{C}}$ with its image in $\mathrm{Aut}_{\mathbb{R}}(\mathfrak{g}^{\mathbb{C}})$ on which G acts by conjugation.

The following special case of a general result on conjugacy classes (see [26, p. 117] and [7]) is of basic use.

LEMMA 2.1.4. — *Let V be a finite-dimensional $\mathbb{R}$-vector space, H a closed reductive algebraic subgroup of* $\mathrm{GL}_{\mathbb{R}}(V)$ *and* $s \in \mathrm{GL}_{\mathbb{R}}(V)$ *an element which normalizes H. Regard H as acting on* $\mathrm{GL}_{\mathbb{R}}(V)$ *by conjugation. Then, for a semi-simple s the orbit $H \cdot s$ is closed.*

COROLLARY 2.1.5. — *The image* $\mathrm{Im}(\eta)$ *is closed in* $\mathrm{Aut}_{\mathbb{R}}(\mathfrak{g}^{\mathbb{C}})$.

Proof. — It is enough to show that $G^{\mathbb{C}}.\tau = \{\mathrm{Ad}(g)\tau\,\mathrm{Ad}(g^{-1}) : g \in G^{\mathbb{C}}\}$ is closed in $\mathrm{Aut}_{\mathbb{C}}(\mathfrak{g}^{\mathbb{C}})$. Since τ is semi-simple and normalizes $G^{\mathbb{C}}$ in this representation, this follows from Lemma 2.1.4. □