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SCHÉMAS EN GROUPES ET IMMEUBLES DES GROUPES EXCEPTIONNELS SUR UN CORPS LOCAL. DEUXIÈME PARTIE : LES GROUPES F_4 ET E_6

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RÉSUMÉ. — Nous obtenons une version explicite de la théorie de Bruhat-Tits pour les groupes exceptionnels des type F_4 ou E_6 sur un corps local. Nous décrivons chaque construction concrètement en termes de réseaux : l'immeuble, les appartements, la structure simpliciale, les schémas en groupes associés.

ABSTRACT. — We give an explicit Bruhat-Tits theory for the exceptional group of type F_4 or E_6 over a local field. We describe every construct concretely in terms of lattices: the building, the apartments, the simplicial structure, and the associated group schemes.

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Introduction

In this sequel to our paper [9], we give an explicit description of the Bruhat-Tits theory [4–8] for a split exceptional group G of type F_4 or E_6 over a local field. More precisely, we give a natural and explicit model of the Bruhat-Tits building $\mathcal{B}(G)$ as a topological space, describe its simplicial structure, the structure of apartments and the associated parahoric group schemes in terms of this model, and discuss the relations among buildings of different groups. We refer the reader to the introduction of [9] (where the case $G = G_2$ was handled) for the goal and the history of this programme.

Many techniques used in this paper have already been developed in [9], or can at least be implicitly found there. However, since the rank of G_2 is very small, a proof in [9] can occasionally be achieved by staring at the Coxeter complex which is an apartment of $\mathcal{B}(G_2)$ (the figure in [9, §9]). Here it is necessary to develop a more systematic approach. We now outline our general strategy for studying the building of a simply connected simple group G over a local field k.

Step 1. Choosing a geometric description of G. — Namely, we realize G as $\mathrm{Aut}(V,T)$, where V is a vector space over k and $T = \{t_i\}$ is a set of tensors on V. Naturally, we prefer to make $\dim V$ small and the description of T economical.

Step 2. Embedding of buildings. — Let $\iota : G \to \mathrm{GL}(V)$ be the natural embedding and show that this extends to a strong descent datum $\iota_* : \mathcal{B}(G) \to \mathcal{B}(\mathrm{GL}(V))$ of the Bruhat-Tits buildings. In general, there may be many choices for ι_*, but in the cases treated in this paper, the choice of ι_* is essentially unique.

Step 3. Determination of the image of ι_*. — This can be achieved using the formalism in [9, §3]. Recall from the fundamental work [6] of Bruhat and Tits that $\mathcal{B}(\mathrm{GL}(V))$ can be identified with the set of norms on V. Hence, determining the image of ι_* amounts to describing $\mathcal{B}(G)$ as the set of norms on V satisfying suitable conditions (expressed in terms of the tensors $\{t_i\}$), and this gives the desired model of $\mathcal{B}(G)$ as a topological space. We remark that the key input for the formalism of [9, §3] is usually an arithmetic result. In [9], this key arithmetic result is the fact that any two maximal orders in the split octonion algebra are isomorphic. Here, the key input is a theorem of Racine [16] that any two distinguished orders in the split simple exceptional Jordan algebra are isomorphic.

Step 4. Making a list of graded lattice chains and their properties. — Recall from [6] that the norms on V are in natural bijection with graded lattice chains in V. For a "standard" closed chamber C on $\mathcal{B}(G)$ and each vertex $v \in C$, one can actually write down the norm $\alpha_v = \iota_*(v)$, and its associated graded lattice chain $(L_\bullet, c)$. The stabilizer in $G(k)$ of the graded lattice chain $(L_\bullet, c)$ is then equal to the stabilizer of the vertex v, and hence is a maximal parahoric subgroup of $G(k)$. In fact, since G is simply-connected in our case, the stabilizer of any member of the lattice chain $L_\bullet$ must already be the maximal parahoric subgroup. This suggests that the graded lattice chain $(L_\bullet, c)$ (and hence the vertex v) can be reconstructed from one particular member $L(v)$ of $L_\bullet$, as a consequence of certain properties that $L(v)$ possesses. Usually, we simply take

$$L(v) = \bigl\{x \in V : \alpha_v(x) \geq 0\bigr\}.$$

By examining the graded lattice chain for each vertex v on C, we make such a list P_v of properties that $L(v)$ satisfies. We distinguish two kinds of properties: (i) the basic numerical invariants of $L(v)$ and its associated graded lattice chain $(L_\bullet, c)$, such as the image of c and the volumes of the members of $L_\bullet$ (see the beginning of §5 for the notion of volume); (ii) other properties, whose description usually involve the tensors $\{t_i\}$.

Step 5. Lattice-theoretic description of the vertices. — By Step 4, we have an injective map

$$\begin{aligned}\{\text{vertices of } \mathcal{B}(G) \text{ conjugate to } v\} &\longrightarrow \{\text{lattices in } V \text{ satisfying property } P_v\},\\ x &\longmapsto L(x),\end{aligned}$$

and we would like to show that it is surjective. This is achieved systematically as follows.

- ▷ Given a lattice $L \subset V$ satisfying P_v, we reconstruct a graded lattice chain $(L_\bullet, c)$ which corresponds to a norm α_L on V.
- ▷ Using the description of $\mathcal{B}(G)$ in Step 3, we check that α_L lies on $\mathcal{B}(G)$. Hence, we can conjugate it to a point in the closed "standard" chamber C using $G(k)$, and we need to show that $\alpha_L = v$.
- ▷ If $\mathcal{A}$ is a "standard" apartment of $\mathcal{B}(G)$ containing C, we identify the subset of $\mathcal{A}$ consisting of those norms α whose associated lattices $L(\alpha)$ satisfy part (i) of P_v. This is practicable and very useful since this subset lies in a lattice M_v in the affine space $\mathcal{A}$. The point α_L thus lie on $M_v \cap C$, which is a finite set.

- ▷ Using part (ii) of P_v, we show that v can be distinguished from other points in $M_v \cap C$.

This gives the desired description of the vertices of $\mathcal{B}(G)$ in terms of certain lattices in V.

Step 6. Determination of the simplicial structure. — We would like to show that if x and y are vertices, then x is incident to y if and only if there is an inclusion relation, say $L(y) \subset L(x)$. From the explicit list of graded lattice chains made in Step 4, such an inclusion relation is easily seen to be necessary, and it remains to show that it is also sufficient. After this is done, we would have a purely lattice-theoretical description of the simplicial complex $\mathcal{B}(G)$.

To prove the expected characterization of incidence relation, we may assume that x and y lie on the "standard" apartment $\mathcal{A}$. If N_{xy} is the number of vertices of type y incident to x, and N'_{xy} is the number of vertices z of type y such that $L(z) \subset L(x)$, then it suffices to show that $N_{xy} = N'_{xy}$. The number N_{xy} can be computed using the theory of Coxeter complexes, whereas the number N'_{xy} can be found with the aid of the computer. Indeed, we first identify the bounded set $B_x = \{z \in \mathcal{A} : L(z) \subset L(x)\}$. The points in B_x which satisfy part (i) of P_y lie in the finite set $B_x \cap M_y$. One can then use the computer to count the points in $B_x \cap M_y$ which satisfy P_y, and show that $N_{xy} = N'_{xy}$.

Step 7. Construction of the Bruhat-Tits schemes. — Let x be a vertex on $\mathcal{B}(G)$. We would like to describe its associated smooth model $\underline{G}_x$ of G over A (the ring of integers of k). In many cases, it can be shown that $\underline{G}_x$ is simply the schematic closure of G in $\mathrm{Aut}(L(x))$. The proof, following the paradigm laid out by Bruhat and Tits [5], relies on detailed analysis of the smoothness of schematic closures of root subgroups. More generally, one can construct the Bruhat-Tits scheme associated to a bounded convex set in an apartment by taking a suitable schematic closure.

It is instructive to compare the above programme to the analogous problem of determining the spherical building of G. In the latter case, we do not have the key formalism developed in [9, §3] and used in Step 3. Also, the apartments are simplicial spheres rather than affine spaces, and hence the geometric tricks in Steps 5 and 6 are not available. Indeed, the remarkable paper [1] of Aschbacher, which gives a description of the spherical building of F_4 or E_6 analogous to the conclusions of Steps 5 and 6, involves very different techniques. Since the spherical building of G (over the residue field of A) can be obtained as the link of a hyperspecial vertex in the Bruhat-Tits building, it is natural to ask whether our results can be used to recover Aschbacher's description of the spherical building of a split group of type F_4 or E_6, at least over a perfect field. We do not pursue this here, but in this connection, it is worth pointing out that this paper relies on [1] only in the proof of Proposition 5.3 where we have used [1, (3.16)].

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1. Cubic Forms and Jordan Algebras

We begin with some generalities on cubic forms and Jordan algebras. Let A be a (unital, commutative and associative) ring and J a projective A-module of finite rank. Let N be a cubic form on J, and t its associated symmetric trilinear form. The cubic form N determines a tensor Q on $J \times J$, characterized by the requirement that:

- ▹ for fixed y, $L_y : x \mapsto Q(x,y)$ is a linear form;
- ▹ for fixed x, $Q_x : y \mapsto Q(x,y)$ is a quadratic form;
- ▹ $N(x+y) - N(x) - N(y) = Q(x,y) + Q(y,x)$.

The 3-tuple (N,Q,t) satisfies

- ▹ the symmetric bilinear form associated to Q_x is $t(x,-,-)$, *i.e.*

$$t(x,y,y) = 2 \cdot Q(x,y),$$

- ▹ $Q(x,x) = 3 \cdot N(x)$,

and is called a regular 3-form in [1].

Let $e \in J$ be such that $N(e) = 1$. Then we obtain a symmetric bilinear T by setting

$$T(x,y) = Q(x,e)Q(y,e) - t(e,x,y).$$

If this symmetric bilinear form is non-degenerate, *i.e.* induces an isomorphism $J \to \operatorname{Hom}_A(J,A)$, then we can define a quadratic map $\#$ on J by the formula

$$T(x^{\#}, y) = Q(y,x).$$

In that case, we set

$$x \times y = (x+y)^{\#} - x^{\#} - y^{\#}.$$

Following Jacobson [12, §2.4], the pair (N,e) is said to be admissible if:

- ▹ T is non-degenerate,
- ▹ the quadratic map $\#$ satisfies $x^{\#\#} = N(x) \cdot x$.

Given an admissible pair (N,e), we have the following useful identities:

$$e^{\#} = e, \tag{1}$$

$$T(x \times y, z) = T(x, y \times z) = t(x,y,z), \tag{2}$$

$$e \times x = T(e,x)e - x. \tag{3}$$