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ON THE PYTHAGORAS NUMBERS OF REAL ANALYTIC SET GERMS

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Abstract. — We show that (i) the Pythagoras number of a real analytic set germ is the supremum of the Pythagoras numbers of the curve germs it contains, and (ii) every real analytic curve germ is contained in a real analytic surface germ with the same Pythagoras number (or Pythagoras number 2 if the curve is Pythagorean). This gives new examples and counterexamples concerning sums of squares and positive semidefinite analytic function germs.

Résumé (*Sur le nombre de Pythagore des germes d'ensembles analytiques réels*)
Nous montrons : (i) que le nombre de Pythagore d'un germe d'ensemble analytique réel est le plus grand des nombres de Pythagore des courbes qu'il contient et (ii) que tout germe de courbe analytique réelle est contenu dans le germe d'une surface analytique réelle ayant le même nombre de Pythagore (ou le nombre 2 si la courbe est pythagoricienne). Cela fournit de nouveaux exemples et contre-exemples à propos des sommes de carrés et des germes de fonctions analytiques semi-définies.

1. Preliminaries and statement of results

The Pythagoras number of a ring A is the smallest integer $p(A) = p \geq 1$ such that any sum of squares of A is a sum of p squares, and $p(A) = +\infty$ if such an

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integer does not exist. This invariant appeals specialists from many different areas, and has a very interesting behaviour in geometric cases; we refer the reader to [4], [7], [21] and [22]. Here we are interested in the important case of real analytic germs, which have been extensively studied in [6], [17], [18], [11], [20], [12], [8], [9], [10].

Let $X \subset \mathbb{R}^n$ be a real analytic set germ and $\mathcal{O}(X)$ its ring of analytic function germs. Since $\mathcal{O}(\mathbb{R}^n)$ is the ring $\mathbb{R}\{x\}$ of convergent power series in $x = (x_1, \ldots, x_n)$, we have $\mathcal{O}(X) = \mathbb{R}\{x\}/\mathcal{J}(X)$, where $\mathcal{J}(X)$ stands for the ideal of analytic function germs vanishing on X. We will discuss the Pythagoras number $p[X] = p(A)$ of the ring $A = \mathcal{O}(X)$.

Clearly, if we have another real analytic set germ $Y \subset X$, then $\mathcal{J}(Y) \supset \mathcal{J}(X)$ and the canonical surjection $\mathcal{O}(X) \to \mathcal{O}(Y)$ gives immediately the inequality $p[Y] \leq p[X]$. This easy remark can be sharpened as follows:

Theorem 1.1. — *The Pythagoras number $p[X]$ of a real analytic set germ X is the supremum of the Pythagoras numbers $p[Y]$ of all real analytic curve germs $Y \subset X$.*

If the germ X is irreducible, the supremum can be restricted to *irreducible* curve germs Y. In general this is not possible:

Example 1.2. — The planar curve germ $Y : (x^2 - y^3)(x^2 + y^3) = 0$ has Pythagoras number 2, while its irreducible components $Y_1 : x^2 - y^3 = 0$ and $Y_2 : x^2 + y^3 = 0$ have both Pythagoras number 1.

Indeed, since $Y \subset \mathbb{R}^2$, we have $p[Y] \leq 2$, and looking at the initial forms of the series involved, one easily checks that $x^2 + y^2$ is not a square mod $(x^2 - y^3)(x^2 + y^3)$. On the other hand, $\mathcal{O}(Y_i) \equiv \mathbb{R}\{t^2, t^3\}$, and this ring consists of all power series without the degree 1 monomial; it follows readily that in this ring every sum of squares has always a square root.

Note that if X itself is a curve germ, Theorem 1.1 is trivial. On the other hand, in the irreducible case the result is quite more precise, as it takes the form of a curve selection lemma:

Theorem 1.3. — *Let X be an irreducible real analytic set germ of dimension ≥ 2, and $Z \subset X$ a semianalytic germ with* $\dim(Z) = \dim(X)$. *Then $p[X]$ is the supremum of the Pythagoras numbers $p[Y]$ of all irreducible curve germs Y such that $Y \setminus \{0\} \subset Z$.*

The condition $Y \setminus \{0\} \subset Z$ means that Z contains both open half-branches of Y, which improves the more typical one half-branch selection; this is important for applications (see [2, VII.4,5]). Summing up, there are two possibilities:

(i) If $\dim(X) = 2$, then $p[X] = p < +\infty$ (see [8]), and we can find a curve germ $Y \subset X$, with $p[Y] = p$. As said above, if X is irreducible, the curve germ Y can be chosen irreducible and anywhere in X.

(ii) If $\dim(X) \geq 3$, then $p[X] = +\infty$ (see [9]), and what happens is that X contains anywhere irreducible curve germs with Pythagoras number arbitrarily large (note that in this case we can always suppose X irreducible). For instance, in $X = \mathbb{R}^3$ we can find monomial curve germs $Y : x_i = t^{m_i}$ with $p[Y] \to +\infty$.

In this statements, *anywhere* means *in any semianalytic set germ of maximal dimension.* We will prove this in Section 2.

In case (ii) we can also find irreducible surface germs $X' \subset X$ with arbitrarily large Pythagoras number $p[X']$. For that, we first find irreducible curve germs $Y_k \subset X$ not contained in the singular locus of X such that $p[Y_k] \to +\infty$; then, by the condition on the singular locus, X contains some irreducible surface germ $X_k \supset Y_k$ (see [2, VII.5.1,VIII.2.5]), so that $p[X_k] \geq p[Y_k]$.

After these results it is only natural to seek for a converse, namely:

Theorem 1.4. — *Let Y be a curve germ. Then there exist a pure surface germ $X \supset Y$ with*

$$p[X] = \begin{cases} 2 & \textit{if } \; p[Y] = 1, \\ p[Y] & \textit{otherwise.} \end{cases}$$

We call a set germ *pure* if its irreducible components have all the same dimension. In fact, if Y is irreducible, X can be found irreducible too. The statement above is the best possible one, since surface germs have Pythagoras number ≥ 2.

The proof of Theorem 1.4 is developed in Section 3. It is interesting to remark here that for irreducible Y, the surface X one obtains is *birational to* $\mathbb{R}^2$, that is, it has a parametrization $x = x(s,t)$ that induces an isomorphism $\mathcal{M}(X) \to \mathbb{R}(\{s,t\})$ between the fields of meromorphic function germs. In particular, although the Pythagoras number $p[X]$ of the domain $\mathcal{O}(X)$ is arbitrary, the Pythagoras number $p(X)$ of its field of fractions $\mathcal{M}(X)$ is always 2: $p(X) = p(\mathcal{M}(X)) = p(\mathbb{R}(\{s,t\})) = 2$

The construction used to prove Theorem 1.4 can be extended to obtain the following *relative version* of the result: *If a surface germ X has some irreducible components of dimension* 1*, then those components embed in a surface germ X' so that $p[X \cup X'] = p[X]$.* The proof of this technical generalization is most predictable, and will not be detailed here.

We also notice that the surface X may well need bigger embedding dimension than the curve Y:

Example 1.5. — Consider the curve germ $Y \subset \mathbb{R}^3$ given by

$$Y : x = t^5, \; y = t^{11}, \; z = t^{18}.$$

The Pythagoras number of this curve germ is $p[Y] = 2$, as one can see after some (not completely straightforward) work using some ideas in [17], [18] and [11]. Consequently, by Theorem 1.4, Y is contained in some surface germ X with $p[X]=2$, but no such a surface germ can be embedded in $\mathbb{R}^3$.

This can be proven by way of contradiction, as we sketch next. Suppose $X\subset\mathbb{R}^3$, defined by an equation $f(x,y,z)=0$ which must have order 2 (otherwise, $x^2+y^2+z^2$ would not be a sum of two squares mod f). As f belongs to $\mathcal{J}(Y)=(z^2-yx^5, zy^2-x^8, y^3-x^3z)$, we can take

$$f=z^2-yx^5+2a(zy^2-x^8)+2b(y^3-x^3z),$$

with $a,b\in\mathbb{R}\{x,y\}$. By the Weiertrass Preparation Theorem, we factorize $f=UP$, where

$$U\in\mathbb{R}\{x,y,z\} \text{ is a unit, and } P=z^2+2B(x,y)z+C(x,y).$$

A small computation gives

$$U(x,y,0)C=f(x,y,0)\in(y^3,x^6),$$
$$2U(x,y,0)B+\frac{\partial U}{\partial z}(x,y,0)C=\frac{\partial f}{\partial z}(x,y,0)\in(y^2,x^3)$$

and after the change of coordinates $v=z+B$, f becomes v^2+C-B^2. Since U is a unit, $U(x,y,0)$ is a unit too, and we deduce

$$C\in(y^3,x^6)\subset(y,x^2)^3,\quad B^2\in(y^2,x^6)^2\subset(y,x^2)^3.$$

Thus our surface germ is now $X: v^2=F,$ where $F=B^2-C\in(y,x^2)^3$. This surface must be equivalent to some surface germ in the list given in [10], but standard singularity theory (see [14, 9.2.12–14]) tells that this is not possible. □

Thus, we can only expect a mild control on the embedding dimension of X. In fact, a careful analysis of the construction in Section 3 will give some bound for that embedding dimension in terms of numerical invariants of Y. For instance, if Y is the curve of the example above, the embedding dimension of X can be lowered to 18!

An interesting consequence of Theorem 1.4 is this:

Corollary 1.6. — *Every integer $p\geq 2$ is the Pythagoras number of a real analytic surface germ.*

Proof. — Indeed, p is the Pythagoras number of some curve germ by [17], and then 1.4 applies. □

We can look more closely at the construction of these surface germs, which give new examples concerning the problem whether every positive semidefinite analytic function germ is a sum of squares of analytic function germs (in short, psd = sos). In fact, given a curve Y with Pythagoras number $p\geq 2$, the surface X with $p[X]=p$ lies in a sandwich $Y\equiv Y\times\{0\}\subset X\subset Y\times\mathbb{R}^d$ for suitable $d\geq 1$. Thus:

(1) Any positive semidefinite function germ on Y, extends to a positive semidefinite function germ on X, and

(2) Every sum of squares on X restricts to a sum of squares on Y.

Consequently, since psd $\neq$ sos for Y, we conclude psd $\neq$ sos for X. In particular, choosing Y with $p[Y] = 2$, we produce a full range of *new examples of real analytic surface germs X with minimal Pythagoras number and psd $\neq$ sos*. These examples include those in [10, Ex.1.5], which correspond to the simplest possible Y's: the planar curves $x^n = y^{n+1}$.

2. Proof of Theorems 1.1 and 1.3

The key result to these theorems is a *curve selection lemma with large tangent space*, which refines [3, Prop. 1] in various ways to fit our situation. We present a different proof which simplifies that of [3] and gives the generalization needed here:

LEMMA 2.1. — *Let $X \subset \mathbb{R}^n$ be a real analytic irreducible germ of dimension ≥ 2, and $Z \subset X$ a semianalytic germ with* $\dim(Z) = \dim(X)$. *Then, for every integer $k \geq 1$ there is a real analytic curve germ $Y \subset X$ such that:*

$$Y \setminus \{0\} \subset Z \quad \textit{and} \quad \mathcal{J}(Y) \subset \mathcal{J}(X) + (x_1, \ldots, x_n)^k.$$

Proof. — For the proof, we can suppose

$$Z = \{f_1 > 0, \ldots, f_r > 0\} \cap X,$$

where $f_1, \ldots, f_r \in \mathbb{R}\{x\}$. Since X is irreducible, the ring $A = \mathcal{O}(X) = \mathbb{R}\{x\}/\mathcal{J}(X)$ is a domain, whose quotient field we denote by K. By the hypothesis on the dimension of Z, there is a total ordering α of K such that $f_1(\alpha) > 0, \ldots, f_r(\alpha) > 0$. Write $(x) = (x_1, \ldots, x_n)$ and let $\mathfrak{m} = (x) \bmod \mathcal{J}(X)$ be the maximal ideal of A. We consider the convex hull V of $\mathbb{R}$ in K with respect to α (see [2, II.3.6]): V is a valuation ring of K with residue field $\mathbb{R}$, and since A is henselian, V dominates A. Now, by Hironaka's resolution of singularities (see [13]), there is a finitely generated regular A-algebra A' such that $f = f_1 \cdots f_r$ has only normal crossings in A'. Furthermore, A' is proper over A, that is, if a valuation ring of K contains A, then it contains A'. Consequently, our valuation ring V dominates some localization B of A'; let $\mathfrak{n}$ denote the maximal ideal of B. By Zariski's Subspace Theorem (see [1, 10.6]), A is a subspace of B (with respect to the adic topologies); consequently:

(2.1.1) *There is an integer $\ell \geq 1$ such that $\mathfrak{n}^\ell \cap A \subset \mathfrak{m}^k$.*

Note also that, since the residue fields of A and V are both $\mathbb{R}$, the residue field of B is also $\mathbb{R}$. By the normal crossings condition on f, we can write

$$f_j = u_j y_1^{m_{j1}} \cdots y_d^{m_{jd}},$$