Bull. Soc. math. France
133 (3), 2005, p. 419–458

FREE DECAY OF SOLUTIONS TO WAVE EQUATIONS ON A CURVED BACKGROUND

by Serge Alinhac

Abstract. — We investigate for which metric g (close to the standard metric g_0) the solutions of the corresponding d'Alembertian behave like free solutions of the standard wave equation. We give rather weak (*i.e.*, non integrable) decay conditions on $g - g_0$; in particular, $g - g_0$ decays like $t^{-\frac{1}{2}-\varepsilon}$ along wave cones.

Résumé (*Décroissance des solutions des équations d'ondes sur un arrière-plan courbe*)
Nous étudions pour quelles métriques g (proches de la métrique standard g_0) les solutions du d'Alembertien pour g se comportent comme des solutions libres de l'équation des ondes standard. Nous proposons des conditions de décroissance assez faibles (*i.e.*, non intégrables) sur $g - g_0$; en particulier, $g - g_0$ décroît comme $t^{-\frac{1}{2}-\varepsilon}$ le long des cônes d'onde.

Introduction

We consider the wave equation L_g associated with a given Lorentzian metric g on $\mathbb{R}_t \times \mathbb{R}^3_x$. Our aim is to answer the question: under which conditions on g do the solutions of $L_g u = 0$ behave like free solutions of the standard wave equation L_0 ? One can of course use the energy method of Klainerman, commuting the standard "Z"-fields with the equation, and putting on g strong enough decay assumptions (relative to the standard metric) to obtain finally a

Texte reçu le 26 juin 2003, révisé le 30 septembre 2003, accepté le 21 janvier 2004

Serge Alinhac, Département de Mathématiques, Université Paris-Sud, 91405 Orsay (France) • *E-mail :* `serge.alinhac@math.u-psud.fr`

2000 Mathematics Subject Classification. — 35L40.

Key words and phrases. — Energy inequality, wave equation, decay of solutions.

BULLETIN DE LA SOCIÉTÉ MATHÉMATIQUE DE FRANCE 0037-9484/2005/419/$ 5.00

control of $|\partial Z^k u|_{L^2}$, which implies in turn, thanks to Klainerman's inequality, the behavior

$$|\partial u| \leq C\big(1+\big|t-|x|\big|\big)^{-\frac{1}{2}}\big(1+t+|x|\big)^{-1}.$$

What we have in mind is to impose as little decay as possible on g, getting close to what seems to be a *critical level.* The framework we choose here is one where a "1D-situation" occurs, in the sense of [2]. This means that we can prove for L_g an energy inequality in which three special derivatives G (the "good" derivatives) are better controlled than in the standard $L^\infty_t L^2_x$-norm: only one "bad" derivative is left. This idea has been used already in [1], where we study the equation

$$\partial_t^2 u - c^2(u)\Delta u = 0.$$

This later work splits essentially into a linear part, where we study the operator $\partial_t^2 - c^2(u)\Delta$, and a nonlinear part which is a bootstrap on certain properties of u. Because of the very special form of the equation, it seemed to us that the treatment of the linear problem involved many miracles which were may be not likely to occur again in a more general case. Also, in this nonlinear problem, u was likely to decay roughly as t^{-1}, implying a similar decay for derivatives of $c(u)$. The general analysis below shows that one can relax this assumption down to an almost $t^{-\frac{1}{2}}$ decay of the metric (relative to the flat metric).

A more precise discussion of these issues will be offered in section 1.4 after our notations, assumptions and results have been stated. Let us just say here that the whole paper is strongly inspired by the geometric techniques of Christodoulou and Klainerman, developed in [4], [3] and also by related work of Klainerman and Sideris [10], Klainerman and Nicolò [8] and Klainerman and Rodnianski [9].

1. Framework and main result

1.1. The general framework. — We work in $\mathbb{R}_t \times \mathbb{R}^3_x$ where

$$x^0 = t, \quad x = (x^1, x^2, x^3), \quad \partial_\alpha = \frac{\partial}{\partial x^\alpha}, \quad r = |x|, \quad r\omega = x, \quad \sigma = \langle r-t\rangle,$$

where here and below we use

$$\langle s\rangle = (1+s^2)^{\frac{1}{2}}.$$

As usual, the greek indices will run from 0 to 3, while the latin one will run only from 1 to 3.

We consider a metric $g = g_0 + \gamma$ which is a (small) perturbation of the standard Minkowski metric g_0 defined by

$$(g_0)_{00} = -1, \quad (g_0)_{ii} = 1, \quad (g_0)_{0i} = 0.$$

The inverse matrix to $g_{\alpha\beta}$ is denoted by $g^{\alpha\beta}$. We will write

$$\langle X, Y\rangle = g(X,Y)$$

and denote by D the connexion associated to g. Recall that for a function a, the gradient of a and the Hessian of a are defined by

$$\nabla a = g^{\alpha\beta}(\partial_\alpha a)\partial_\beta, \quad \nabla^2 a(X,Y) = XYa - (D_X Y)a.$$

We denote by L_0 the d'Alembertian associated to g_0 (the standard wave equation), and by

$$L_g u = g^{\alpha\beta}\nabla^2 u_{\alpha\beta}$$

the d'Alembertian associated to g. We assume

$$g^{00} = -1, \quad g^{0i}(x,t)\omega_i = 0,$$

and define

$$T = -\nabla t = \partial_t - g^{0i}\partial_i, \quad N = \frac{\nabla r}{|\nabla r|}, \quad L = T + N, \quad L_1 = T - N.$$

Note that our assumption $g^{0i}\omega_i = 0$ allows us to express $T - \partial_t$ and $N - c\partial_r$ using the standard rotations, a fact which will be important later on. As shown in [2], we have the easy properties

$$\langle T,T\rangle = -1, \quad T(r) = 0 = \langle N,T\rangle, \quad D_T T = 0,$$
$$\langle L,L\rangle = 0 = \langle L_1,L_1\rangle, \quad \langle L,L_1\rangle = -2.$$

We use the frame

$$e_1, e_2, L_1, L,$$

where the e_i form an orthonormal basis on the standard spheres $t = t_0$, $r = r_0$.

Three quantities play an important role in the following:

- the radial sound speed c defined by

$$c = |\nabla r|, \quad c^2 = g^{ij}\omega_i\omega_j,$$

- the second fundamental form k of the hypersurfaces $t = \text{Constant}$,

$$k(X,Y) = \langle D_X T, Y\rangle, \quad k_{ij} = \frac{1}{2}g^{0\alpha}(\partial_i g_{\alpha j} + \partial_j g_{\alpha i} - \partial_\alpha g_{ij}),$$

- the second fundamental form of the standard spheres $t = t_0$, $r = r_0$ in $\{t = t_0\}$

$$\theta(e,e') = \langle D_e N, e'\rangle,$$

where e and e' are tangent to the sphere.

We denote by $\bar{k}$ and $\bar{\theta}$ the traces of theses forms

$$\bar{k} = k(N,N) + k(e_1,e_1) + k(e_2,e_2), \quad \bar{\theta} = \theta(e_1,e_1) + \theta(e_2,e_2).$$

In the frame (e_i, L_1, L), the d'Alembertian L_g is

$$L_g = -LL_1 + \Delta_S - \bar{k}T + (k_{NN} + \bar{\theta})N + \sum_{i=1,2}\Big(2k_{iN} - \frac{e_i(c)}{c}\Big)e_i,$$

where Δ_S is the Laplacian on the standard spheres corresponding to the restriction of g to these spheres. Finally, we recall the definitions of the standard fields

$$R_i = (x \wedge \partial)_i, \quad S = t\partial_t + r\partial_r.$$

1.2. Assumptions on the metric. — The behavior of the metric and of the solution will be discussed in terms of the two parameters

$$\sigma = \big(1 + (r-t)^2\big)^{\frac{1}{2}}, \quad 1 + t + r.$$

Because of this, we distinguish three zones I, II and III, respectively defined by

$$r \le \tfrac{1}{2}(1+t), \quad \tfrac{1}{2}(1+t) \le r \le \tfrac{3}{2}(1+t), \quad r \ge \tfrac{3}{2}(1+t),$$

which we also call "interior", "middle zone" and "exterior". The reason for using these parameters is that in nonlinear applications, the coefficients γ will be functions of u or ∂u, and their behavior has to be discussed in the same terms as the behavior of u.

The time decay of certain quantities will be measured using a smooth increasing $\phi = \phi(t) > 0$ such that

$$(1.2)_{\mathrm{a}} \qquad \phi' > 0, \quad (1+t)\phi' \in S^0, \quad \frac{\phi''}{\phi'} \in S^{-1},$$

$$(1.2)_{\mathrm{b}} \qquad \forall \epsilon > 0, \quad \phi(t) \le C_\epsilon + \epsilon \log(1+t).$$

Here, S^m denotes symbols of order m, that is, smooth functions $s(t)$ satisfying

$$\big|s^{(k)}(t)\big| \le C_k \langle t\rangle^{m-k}, \quad k \in \mathbb{N}.$$

In [1], we take $\phi(t) = \epsilon \log(1+t)$. The "free case" corresponds to the choice ϕ' integrable. It seemed however relevant to us to incorporate in the present paper certain decay patterns which played in important role in [1].

There are three groups of assumptions on the metric:

- *General low decay.* — For some $\mu > \frac{1}{2}$, and all k,

$$|\Gamma^k \gamma^{\alpha\beta}| \le \gamma_0\, \sigma^{\frac{1}{2}} (1+t+r)^{-\mu}, \quad |\Gamma^k\, \partial\gamma^{\alpha\beta}| \le \gamma_0\, \sigma^{-\frac{1}{2}} (1+t+r)^{-\mu}.$$

Here, Γ^k means any product of k fields Γ among R_i, S or $\sigma^\mu \partial_\alpha$. In zones I or III, it is enough to take Γ among R_i, S or ∂_α.

- *Special high decay.* — For the quantities $\bar{k}$, $\bar{\theta}$ and c, we have *in the middle zone* the high decay

$$|\Gamma^k \bar{k}| \le \gamma_0\, \sigma^{-\frac{1}{2}} (1+t)^{-1}, \quad |\Gamma^k \bar{\theta}| \le \gamma_0 (1+t)^{-1},$$
$$|1-c| \le \gamma_0\, \sigma^{\frac{1}{2}} \phi', \quad |\partial c| \le \gamma_0\, \sigma^{-\frac{1}{2}} \phi',$$
$$|\Gamma^{k+1} c| \le \gamma_0\, \sigma^{\frac{1}{2}} \phi' \mathrm{e}^{C\phi}, \quad |\Gamma^k \partial c| \le \gamma_0\, \sigma^{-\frac{1}{2}} \phi' \mathrm{e}^{C\phi}.$$

- *Technical interior assumption.* — In the interior, we assume $r|\bar{\theta}| \le C$.

Remark. — One can observe in the assumptions above that whenever a quantity is bounded by $*\sigma^{\frac{1}{2}}$, its gradient is bounded by $*\sigma^{-\frac{1}{2}}$. This "homogeneity" is important and occurs naturally in the context of nonlinear equations, where energy methods and Klainerman's formula give no better than a $\sigma^{-\frac{1}{2}}$ control of ∂u (see Introduction); this does not allow in general anything better than u controlled by $\sigma^{\frac{1}{2}}$. We postpone to section 1.4 the discussion of these assumptions.

1.3. Main result. — Let u be the solution of the Cauchy problem

$$L_g u = 0, \quad u(x,0) = u_0(x), \quad (\partial_t u)(x,0) = u_1(x).$$

Assume the following decay on the smooth real functions u_0, u_1

$$\forall\alpha, \forall\beta,\ |\alpha| \leq |\beta|, \quad x^\alpha \partial_x^\beta u_i \in L^2, \quad i = 1, 2.$$

We have then the following "free" decay property.

Theorem. — *For γ_0 small enough and $r \geq \frac{1}{2}(1+t)$, we have*

$$|\partial u| \leq C\sigma^{-\frac{1}{2}}(1+t+r)^{-1}\mathrm{e}^{C\phi}$$

for some $C > 0$.

Remark 1. — The "free decay" result annouced in the title is obtained by choosing ϕ' integrable, in which case ϕ is bounded and so is $\mathrm{e}^{C\phi}$.

Remark 2. — There is little doubt that the same estimate holds also for $\partial Z^k u$, where $Z = R_i$, $Z = S$ or $Z = \partial_\alpha$. This can be proved using the "hat-calculus" of section 9; we dropped the proof of these additionnal estimates to make the paper a little lighter, if possible.

We did not attempt here to give a poor estimate in the interior zone; getting a good one there (without using the hyperbolic rotations) is a real difficulty, which has been completely skipped in [8] for instance, where the authors work only outside the interior zone. One can may be hope for some extension of the inequality proved in [6] for the wave equation, which displays an improved interior behavior of ∂u.

1.4. Discussion of the method of the proof, of the assumptions, and plan of the paper. — a) The method of proof uses energy inequalities for L_g. In the litterature, there are essentially two approaches:

i) One can use a conformal energy inequality (see [5]), which gives a control of $R_i u$, Su and $H_i u$, with $H_i = t\partial_i + x_i\partial_t$. This is the approach of [7], [8] and [9]. This is enough to get some decay on u, but not quite the precise t^{-1} decay we want (see [7]).