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BRAIDS AND SIGNATURES

by Jean-Marc Gambaudo & Étienne Ghys

Abstract. — A braid defines a link which has a signature. This defines a map from the braid group to the integers which is not a homomorphism. We relate the homomorphism defect of this map to Meyer cocycle and Maslov class. We give some information about the global geometry of the gordian metric space.

Résumé (*Tresses et signatures*). — Une tresse définit un entrelacs qui possède une signature. Ceci définit une application du groupe des tresses vers les entiers qui n'est pas un homomorphisme. Nous relions le défaut d'homomorphisme de cette application au cocycle de Meyer et à la classe de Maslov. Nous donnons quelques informations sur la géométrie globale de l'espace métrique gordien.

1. Introduction

This paper deals with the interaction of the following two standard constructions in knot theory.

• *Braids and closed braids* [4], [12].

In the plane $\mathbf{R}^2$, we consider the sequence of points $x_i^0 = (i, 0)$ (for $i = 1, 2, \ldots$) and we denote by $\mathbf{D}^2(r)$ the disc of radius r, centered at the origin.

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In the space X_n of n-tuples of *distinct* points of $\mathbf{D}^2(n+\frac{1}{2})$, we consider the equivalence relation that identifies two n-tuples if one is obtained from the other by a permutation of the indices. We denote by $\widetilde{X}_n$ the quotient space and by $\pi_n : X_n \to \widetilde{X}_n$ the natural projection. The fundamental group of $\widetilde{X}_n$, based at $\pi_n(x_1^0, x_2^0, \dots, x_n^0)$, is called the n-th *Artin braid group* and is denoted by $\mathbf{B}_n$; its elements are called *braids*. Any braid γ in $\mathbf{B}_n$ is represented by a path $t \in [0,1] \mapsto (x_1^t, x_2^t, \dots, x_n^t) \in X_n$, *i.e.* by a system of n disjoint arcs $t \mapsto (t, x_i^t)$ in the cylinder $[0,1] \times \mathbf{D}^2(n+\frac{1}{2})$, such that $\pi_n(x_1^1, x_2^1, \dots, x_n^1) = \pi_n(x_1^0, x_2^0, \dots, x_n^0)$.

The identification $(x,0) \approx (x,1)$ for all x in $\mathbf{D}^2(n+\frac{1}{2})$ produces a finite collection of simple closed oriented curves in the solid torus $\mathbf{R}/\mathbf{Z} \times \mathbf{D}^2(n+\frac{1}{2})$, images of the arcs $t \mapsto (t, x_i^t)$. The usual embedding of the solid torus in 3-space $\mathbf{R}^3$ and the compactification of $\mathbf{R}^3$ with a point at infinity, allow us to associate with any braid γ an *oriented link i.e.* a collection of disjoint embeddings of an oriented circle in the 3-sphere $\mathbf{S}^3$, called the *closed braid* associated with γ, and denoted by $\widehat{\gamma}$ (see Figure 1).

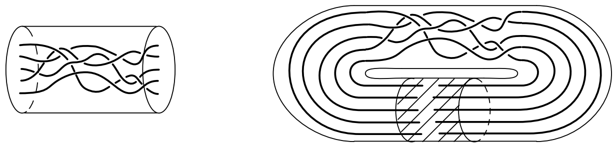


FIGURE 1. Closure of a braid

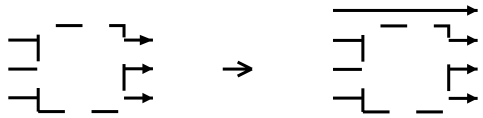
- *Signature of links* [7], [11].

Let $\lambda \subset \mathbf{S}^3$ be an oriented link in $\mathbf{S}^3$ and let us choose a *Seifert surface*: an oriented surface S_λ embedded in $\mathbf{S}^3$ whose oriented boundary is λ. The first homology group $H_1(S_\lambda; \mathbf{Z})$ is equipped with a bilinear *Seifert form* B in the following way. If x and y are two oriented closed curves on S_λ, one defines $B(x,y)$ as the linking number of x with a curve $y^\star$ obtained from y by pushing y a little away from S_λ along the positive direction transverse to S_λ. Clearly $B(x,y)$ only depends on the homology classes of x and y on S_λ. Turning B into a symmetric bilinear form

$$\widetilde{B}(x,y) = B(x,y) + B(y,x)$$

and tensoring by $\mathbf{R}$, we get a symmetric bilinear form on the vector space $H_1(S_\lambda; \mathbf{R})$. It turns out that the signature of this symmetric form is independent of the choice of the Seifert surface: it is the *signature* of the oriented link λ, denoted $\operatorname{sign}(\lambda) \in \mathbf{Z}$. For definiteness, we recall that the signature of a quadratic form is the number of $+$ signs minus the number of $-$ signs in an orthogonal basis.

The notion of signature of an oriented link can be generalized as follows. Tensoring by $\mathbf{C}$, we get a bilinear form on the vector space $H_1(S_\lambda; \mathbf{C})$. Consider



$\alpha \in \mathbf{B}_3$ $\qquad$ $i_3(\alpha) \in \mathbf{B}_4$

FIGURE 2. Adding a trivial strand

a complex number $\omega \neq 1$ (usually chosen as a root of unity) and the hermitian form

$$\widetilde{B}_\omega(x,y) = (1-\omega)B(x,\overline{y}) + (1-\overline{\omega})B(\overline{y},x).$$

The signature of this hermitian form is again independent of the choice of the Seifert surface: it is the ω-*signature*, $\operatorname{sign}_\omega(\lambda) \in \mathbf{Z}$, of the oriented link λ. In the case $\omega = -1$, we recover the signature of the oriented link.

There is a natural sequence of embeddings $\mathbf{B}_1 \subset \mathbf{B}_2 \subset \cdots \subset \mathbf{B}_n \subset \cdots$ of the braid groups. The embedding i_n of $\mathbf{B}_n$ in $\mathbf{B}_{n+1}$ amounts to adding an additional "trivial" strand (see Figure 2). The union of this infinite chain of groups is the *infinite braid group* $\mathbf{B}_\infty$. Note that

$$\operatorname{sign}_\omega(\alpha) = \operatorname{sign}_\omega\bigl(i_n(\alpha)\bigr)$$

since a Seifert surface for $\widehat{i_n(\alpha)}$ is obtained from a Seifert surface for $\widehat{\alpha}$ by adding a disjoint disc. Therefore, the function $\operatorname{sign}_\omega(\widehat{\alpha})$ is well defined on $\mathbf{B}_\infty$.

Combining these constructions, for each ω, we get a map from $\mathbf{B}_\infty$ to $\mathbf{Z}$ which associates with a braid γ the signature $\operatorname{sign}_\omega(\widehat{\gamma})$. We are now in a position to raise the question:

Given two braids α and β in the braid group $\mathbf{B}_\infty$ and a root of unity ω, what can be said about the quantity

$$\operatorname{sign}_\omega(\widehat{\alpha\cdot\beta}) - \operatorname{sign}_\omega(\widehat{\alpha}) - \operatorname{sign}_\omega(\widehat{\beta})?$$

We give an explicit answer to this question. Before giving a precise statement we need to recall another construction which again is standard in low dimensional topology.

- *The Burau representation and the Meyer cocycle.*

Burau defined an explicit linear representation of $\mathbf{B}_n$ in $\mathbf{GL}(n-1,\mathbf{Z}[t,t^{-1}])$, where t denotes some indeterminate. These representations combine to a representation of $\mathbf{B}_\infty$ in the ascending union $\mathbf{GL}(\infty,\mathbf{Z}[t,t^{-1}])$ of the groups $\mathbf{GL}(n-1,\mathbf{Z}[t,t^{-1}])$.

If one specializes t as a complex number ω, we get a linear representation $\mathcal{B}_\omega$ in $\mathbf{GL}(\infty,\mathbf{C})$. In [13], Squier shows that if ω is a complex number of modulus 1, the image of $\mathcal{B}_\omega$ is contained in the unitary group of some non degenerate hermitian form. Since the imaginary part of such a hermitian form is a symplectic form, the Burau representation provides symplectic representations:

$$\mathcal{B}_\omega : \mathbf{B}_\infty \longrightarrow \mathbf{Sp}(\infty, \mathbf{R})$$

where we denote by $\mathbf{Sp}(\infty, \mathbf{R})$ the ascending union of the symplectic groups $\mathbf{Sp}(2g, \mathbf{R})$ (consisting of the symplectic automorphisms γ of $\mathbf{R}^\infty$ which are the identity on all vectors of the canonical basis of $\mathbf{R}^\infty$ except for a finite number of them).

In Section 2, we shall give more information concerning this *Burau-Squier representation*. We shall give explicit formulas for the symplectic form and a topological interpretation which (hopefully) will shed some light on the symplectic nature of the Burau representation, originally introduced by Squier in purely algebraic terms.

The symplectic group $\mathbf{Sp}(2g, \mathbf{R})$ is not simply connected; its universal cover $\widetilde{\mathbf{Sp}}(2g, \mathbf{R})$ defines a central extension

$$0 \to \mathbf{Z} \longrightarrow \widetilde{\mathbf{Sp}}(2g, \mathbf{R}) \longrightarrow \mathbf{Sp}(2g, \mathbf{R}) \to 1.$$

This determines a cohomology class in $H^2(\mathbf{Sp}(2g, \mathbf{R}); \mathbf{Z})$, called the *Maslov* class. It turns out that 4 times the Maslov class can be represented explicitly by an integral valued *Meyer cocycle* which is invariant by conjugation. We shall give more motivation and background for this cocycle in Subsection 3.2 but for the time being we only mention the following "computational" definition (see [9]):

Let γ_1 and γ_2 be two elements in $\mathbf{Sp}(2g, \mathbf{R})$ and denote by E_{γ_1,γ_2} the intersection of the images of $\gamma_1^{-1} - \mathrm{id}$ and $\gamma_2 - \mathrm{id}$. If e belongs to E_{γ_1,γ_2}, choose two vectors v_1 and v_2 such that $e = \gamma_1^{-1}(v_1) - v_1 = v_2 - \gamma_2(v_2)$ and define

$$q_{\gamma_1,\gamma_2}(e) = \Omega(v_1 + v_2, e)$$

where Ω denotes the standard symplectic form on $\mathbf{R}^{2g}$. One checks easily that $q_{\gamma_1,\gamma_2}(e)$ is independent of the choices of v_1 and v_2 and defines a quadratic form on E_{γ_1,γ_2}. By definition, the evaluation of the Meyer cocycle on the pair (γ_1, γ_2), denoted $\mathrm{Meyer}(\gamma_1, \gamma_2)$, is the signature of this quadratic form. Observe that the Meyer cocycle can be coherently defined for elements in $\mathbf{Sp}(\infty, \mathbf{R})$. In other words, if γ_1 and γ_2 are two elements in $\mathbf{Sp}(2g, \mathbf{R})$ seen as elements γ_1' and γ_2' of $\mathbf{Sp}(2g + 2, \mathbf{R})$, the values of $\mathrm{Meyer}(\gamma_1, \gamma_2)$ and $\mathrm{Meyer}(\gamma_1', \gamma_2')$ coincide.

We hope that this mysterious definition will become crystal clear in Section 3.2.

We now state the main result of this paper:

THEOREM A. — *Let α and β two braids in $\mathbf{B}_\infty$ and $\omega \neq 1$ a root of unity. Then:*

$$\mathrm{sign}_\omega(\widehat{\alpha \cdot \beta}) - \mathrm{sign}_\omega(\widehat{\alpha}) - \mathrm{sign}_\omega(\widehat{\beta}) = -\,\mathrm{Meyer}\big(\mathcal{B}_\omega(\alpha), \mathcal{B}_\omega(\beta)\big).$$

REMARK 1. — Since the Meyer cocycle evaluated on $\mathbf{Sp}(2g, \mathbf{R})$ is the signature of a quadratic form on a vector space with dimension smaller that $2g$, it follows easily from the definition of the Burau-Squier representation that, for any positive integer n, and any pair of braids α and β in $\mathbf{B}_n$, we have

$$\left|\operatorname{sign}_\omega(\widehat{\alpha \cdot \beta}) - \operatorname{sign}_\omega(w\widehat{\alpha}) - \operatorname{sign}_\omega(\widehat{\beta})\right| \leq 2n.$$

Thus, for any positive integer n, the map $\alpha \in \mathbf{B}_n \mapsto \operatorname{sign}_\omega(\widehat{\beta}) \in \mathbf{R}$ is a *quasimorphism*. A direct proof of this result can be found in [5] where the authors use this property to construct non trivial quasimorphisms on the group of area preserving diffeomorphisms of the 2-sphere.

REMARK 2. — The Artin braid group $\mathbf{B}_n$ has a standard presentation in terms of generators $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ and relations

$$\sigma_i \cdot \sigma_j = \sigma_j \cdot \sigma_i, \quad \sigma_i \cdot \sigma_{i+1} \cdot \sigma_i = \sigma_{i+1} \cdot \sigma_i \cdot \sigma_{i+1};$$

for all i, j in $\{1, \ldots, n-1\}$ satisfying $|i-j| \geq 2$. See for instance [11].

The closed braids $\widehat{\sigma}_i$ are trivial links. Given a braid β in $\mathbf{B}_n$ which reads $\beta = \sigma_{i_1} \cdots \sigma_{i_\ell}$, we have:

$$\operatorname{sign}_\omega(\widehat{\beta}) = -\sum_{j=2}^{j=\ell} \operatorname{Meyer}\big(\mathcal{B}_\omega(\sigma_{i_1} \cdots \sigma_{i_{\ell-1}}), \mathcal{B}_\omega(\sigma_{i_\ell})\big).$$

This last formula is actually very easy to use for numerical computations since the matrices $\mathcal{B}_\omega(\sigma_{i_\ell})$ are sparse (as we shall see in Section 2).

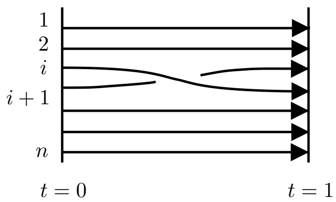


FIGURE 3. The braid σ_i

REMARK 3. — As a trivial illustration of Theorem A, consider the case $n=2$. The image of $\mathcal{B}_{-1}(\mathbf{B}_2)$ is contained in $\mathbf{Sp}(2, \mathbf{R}) = \mathbf{SL}(2, \mathbf{R})$ and it is not difficult to see that, up to conjugacy, $\mathcal{B}_{-1}(\sigma_1)$ is the matrix $\left(\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right)$. Evaluating the Meyer cocycle on 2×2 unipotent matrices is very easy so that one can compute $\operatorname{sign}(\sigma_1^\ell)$ using the method explained in Remark 2. The reader will find immediately the value $1-\ell$ for $\ell \geq 1$. Of course, one can also compute this signature using an explicit Seifert surface for this elementary braid (see [11]).