Bull. Soc. math. France
134 (1), 2006, p. 119–163

ASYMPTOTIC LAWS FOR GEODESIC HOMOLOGY ON HYPERBOLIC MANIFOLDS WITH CUSPS

BY MARTINE BABILLOT & MARC PEIGNÉ

To Martine's memory.
The redaction of this paper has been overshadowed by Martine's death in July 2003. All the main ideas were worked out together, I have done my best to finish this paper. M. Peigné

ABSTRACT. — We consider a large class of non compact hyperbolic manifolds $M = \mathbb{H}^n/\Gamma$ with cusps and we prove that the winding process (Y_t) generated by a closed 1-form supported on a neighborhood of a cusp $\mathcal{C}$, satisfies a limit theorem, with an asymptotic stable law and a renormalising factor depending only on the rank of the cusp $\mathcal{C}$ and the Poincaré exponent δ of Γ. No assumption on the value of δ is required and this theorem generalises previous results due to Y. Guivarc'h, Y. Le Jan, J. Franchi and N. Enriquez.

Texte reçu le 20 septembre 2004, accepté le 1er février 2005.

MARTINE BABILLOT, MAPMO, Fédération Poisson, Université d'Orléans, BP 6759, 45067 OrlŐéans Cedex (France).
MARC PEIGNÉ, Laboratoire de Mathématiques et Physique Théorique, Fédération Poisson, Université François Rabelais de Tours, Parc de Grandmont, 37200 Tours (France).
E-mail : peigne@univ-tours.fr

2000 Mathematics Subject Classification. — 58F17, 58F20, 20H10.

Key words and phrases. — Geodesic flow, asymptotic winding, hyperbolic manifolds, central limit theorem, stable law, transfer operator.

Résumé (*Lois stables et flot géodésique sur des variétés non compactes à courbure négative*)

Nous considérons une large classe de variétés hyperboliques non-compactes $M = \mathbb{H}^n/\Gamma$ possédant des *cusps* et nous démontrons que le processus (Y_t) engendré par une forme fermée portée par un voisinage d'un cusp $\mathcal{C}$ converge en loi vers une loi stable ; la loi limite et le facteur de renormalisation dépendent de la nature du cusp et de l'exposant de Poincaré δ du groupe Γ. Aucune restriction sur la valeur de δ n'est imposée et cet article généralise ainsi toute une série de résultats dus à Y. Guivarc'h, Y. Le Jan, J. Franchi et N. Enriquez.

I. Introduction.

Sinai's observation that "dynamical systems generated by geodesic flows of negatively curved manifolds have a structure analogous to that off dynamical systems generated by stochastic processes" lead him to the proof of a Central Limit Theorem for certain additive functionals of the geodesic flow on a compact hyperbolic manifold M. A typical example of such a functional, due to Gel'fand and Pyateckii-Shapiro [15], is the "winding process", that is the process generated by a closed 1-form on M.

Consider now a (not necessarily compact) hyperbolic manifold $M = \mathbb{H}/\Gamma$ where Γ is a torsion-free group acting isometrically and properly discontinuously on the n-dimensional hyperbolic space $\mathbb{H}$. When Γ is geometrically finite, the manifold M is the union of a compact core and finitely many ends, some of which are funnels, and the other ones are cusps. If a geodesic enters a funnel, it goes to infinity without ever returning to the compact core. On the contrary, a typical geodesic entering a cusp does come back in the compact core infinitely often and thus belongs to the non-wandering set of the geodesic flow. These excursions in the cusps are related to diophantine approximation in number theory when Γ is arithmetic (see [29], [36]), and in general, they describe how well can be approximated boundary points by parabolic ones (see [35]).

From the dynamical point of view, these excursions are now responsible for the non-uniform hyperbolicity of the geodesic flow, and it is therefore unclear which stochastic properties still hold. In this paper, we shall study the winding process around a cusp $\mathcal{C}$ of the manifold. More precisely, we fix a 1-form ω supported and closed on a neighborhood of $\mathcal{C}$. For a vector v in the unit tangent bundle T^1M of M, denote by $[v_0, v_t]$ the geodesic path of length t on the geodesic starting at v. We get an additive functional of the geodesic flow by considering the process

$$Y_t(v) = \int_{[v_0, v_t]} \omega$$

and we are interested in the stochastic behavior of (Y_t) with respect to some invariant measure m of the geodesic flow. In particular, we shall say that (Y_t, m) *satisfies a limit theorem with renormalising factor* $d(t)$ *if there exists a probability measure* π *on* $\mathbb{R}$ *such that for every real number* a

$$m\Big\{v \in T^1M\,;\ \frac{1}{d(t)}Y_t(v) \geq a\Big\} \text{ converges to } \pi(a, +\infty) \text{ as } t \to \infty.$$

One gets the classical Central Limit Theorem when $d(t) = \sqrt{t}$ and π is the Gauss distribution. Otherwise, one looks either for $d(t) = \sqrt{t \log t}$ and π the Gauss distribution, or for $d(t)$ of the form $t^{1/\alpha}$ for some $\alpha \in \,]0, 2[$, and π a stable law of index α.

For finite volume hyperbolic manifolds, the process (Y_t) is known to satisfy a limit theorem with respect to the Liouville measure m. The case of the modular surface $M = \mathbb{H}^2/\operatorname{SL}(2, \mathbb{Z})$ has been worked out by Y. Guivarc'h and Y. Le Jan [18]: the winding of a typical geodesic satisfies a limit theorem with renormalisation factor t and a Cauchy limit law ($\alpha = 1$ in this case). Further works rely on the comparaison between Brownian paths and geodesics, a method introduced by Le Jan [25]. This was used by Enriquez & Le Jan [13] to extend the previous result to any hyperbolic surface and by Franchi [14] for 3-dimensional manifolds; in this last case, the normalising factor becomes $\sqrt{t \log t}$, with a normal limit law. In higher dimension, the Central Limit Theorem holds since in this case the form ω is square integrable with respect to the Liouville measure.

For hyperbolic manifolds with infinite volume, the asymptotic behavior of (Y_t) is still quite open. In this context, the natural probability measure to look at is the Patterson-Sullivan measure m on T^1M when it is finite, since in this case it is the unique measure of maximal entropy; in particular, it gives 0-measure to the wandering set of the geodesic flow, and coincides with the Liouville measure when M has finite volume. Let us call a cusp *neutral* if (Y_t, m) satisfies the Central Limit Theorem, and *influential* if (Y_t, m) satisfies a limit theorem with renormalising factor $d(t) \gg \sqrt{t}$ as $t \to +\infty$. From the series of work mentioned above, we see that for finite volume manifolds, all cusps are influential in dimension 2 and 3, and become neutral in higher dimension.

Our main observation here will be that *this dichotomy on the dimension does not hold anymore* for general hyperbolic manifolds. For a specific class of manifolds, we discover that the main role is played by the *rank* of the cusp $\mathcal{C}$, that is the rank of a maximal free abelian subgroup contained in its fundamental group $P = \pi_1(\mathcal{C})$.

Our main result concerns a restrictive class of Kleinian groups. Let us first introduce a definition: we say that finitely many Kleinian groups $\Gamma_1, \dots, \Gamma_L$ are in *Schottky position* if there exist non-empty disjoint closed sets $F_1, \dots, F_L$ in the boundary $\mathcal{S}^{n-1}$ of $\mathbb{H}^n$ such that $\Gamma_i^*(\mathcal{S}^{n-1} - F_i) \subset F_i$ for any $i \in \{1, \dots, L\}$

(where the notation Γ_i^* stands for $\Gamma_i - \{\mathrm{Id}\}$). The group Γ generated by P and G is called *the Schottky product of P and G*, and P is called a *Schottky factor of* Γ.

We will say that a Schottky product group Γ satisfies the *critical gap hypothesis* if its Poincaré exponent δ is strictly greater than the one of each subgroup Γ_i. For such a group Γ, the Patterson Sullivan measure m on $T^1(\mathbb{H}^n/\Gamma)$ is finite (see [30]), and it is the unique measure of maximal entropy for the geodesic flow restricted to its non wandering set (see [27]); it will thus be the initial distribution for the processes we will consider since, roughly speaking, it caries most of the information about the stochastic behavior of the geodesic flow.

Note that classical Schottky groups (see for instance [26] for a definition) are Schottky products, with each $\Gamma_i \simeq \mathbb{Z}$ and that the critical gap hypothesis is automatically satisfied in this case [3].

We can now state the

MAIN THEOREM. — *Let Γ be a Schottky product of subgroups $\Gamma_1, \dots, \Gamma_L$ of* $\mathrm{Iso}(\mathbb{H}^n)$*, satisfying the critical gap hypothesis. Assume that one of the Schottky factors of Γ is a parabolic group P of rank k and denote by $\mathcal{C}$ the cusp of $\mathbb{H}^n/\Gamma$ associated with P. Let m be the Patterson-Sullivan measure on $T^1(\mathbb{H}^n/\Gamma)$.*

For a closed 1-form ω supported on a neighbourhood of $\mathcal{C}$, the corresponding process (Y_t, m) satisfies a limit theorem. The renormalising factor and the limit law depend on the values of the parameter $\alpha := 2\delta - k$ as follows:

- *if $\alpha < 2$, $d(t) = t^{1/\alpha}$ and the limit law is a stable law of index α;*
- *if $\alpha = 2$, $d(t) = \sqrt{t \log t}$ and the limit law is a normal law;*
- *if $\alpha > 2$, $d(t) = \sqrt{t}$ and the limit law is a normal law.*

Thus, we see that a cusp becomes influential if its rank is sufficiently big with respect to the Hausdorff dimension of the limit set ($k \geq 2\delta - 2$). It should be noted that this condition extends the previous dichotomy for finite volume manifolds. Indeed, in this case, all cusps have maximal rank $k = n - 1$ and the limit set is the whole sphere $\mathcal{S}^{n-1}$; thus, writing $n - 1 \geq 2(n-1) - 2$, we recover the previous condition $n \leq 3$.

Observe that if $\overline{k}$ is the maximal rank of the cusps in M, only cusps of rank $\overline{k}$ and $\overline{k} - 1$ may become influential since, by Beardon's result [3], one knows that δ always satisfies the inequality $2\delta > \overline{k}$.

We believe that this result might be true in particular for all hyperbolic manifolds whose Patterson-Sullivan measure is finite and in particular for all geometrically finite manifolds; this is partly confirmed by a recent result of Enriquez, Franchi & Le Jan [12] who study the case of a manifold with a cusp of rank $n - 1$ under the additional assumption that $2\delta - (n - 1) > 1$.

To prove the Main Theorem, we first have to establish a classical Central Limit Therorem for this class of manifolds. This result can be stated in fact in the case of variable pinched curvature, we have the:

THEOREM III.5. — *Let X be a Hadamard manifold of pinched strictly negative curvature and $\Gamma = \Gamma_1 * \cdots * \Gamma_L$ be a Schottky product of Kleinian groups acting on X and satisfying the critical gap hypothesis. Let $M = X/\Gamma$ be the quotient manifold and m the Patterson-Sullivan probability measure on T^1M. For any bounded and Hölder function $\Phi : T^1M \to \mathbb{R}$, the quantity*

$$\int_{T^1M} \frac{1}{t}\Big(\int_0^t \big(\Phi(g_s v) - m(\Phi)\big)\,\mathrm{d}s\Big)^2 m(\mathrm{d}v)$$

converges to a constant σ_Φ^2. One has $\sigma_\Phi^2 = 0$ if and only if Φ is a coboundary, i.e. if Φ is the derivate in the direction of the flow of some Borel function defined on T^1M.

- *When $\sigma_\Phi^2 \neq 0$, the process $X_t(v) = \int_0^t \Phi(g_s v)\mathrm{d}s$ satisfies the Central Limit Theorem: for any $a \in \mathbb{R}$*

$$m\Big\{v : \frac{X_t(v) - tm(\Phi)}{\sigma_\Phi \sqrt{t}} > a\Big\} \longrightarrow \pi\big([a, +\infty[\big)$$

when $t \to +\infty$, where π is a standard Gaussian law $\mathcal{N}(0,1)$ on $\mathbb{R}$.

- *When $\sigma_\Phi^2 = 0$, the process $((X_t - tm(\Phi))/\sqrt{t})_t$ tends to 0 in probability.*

This theorem extends many previous results; let us cite for instance [34] (without the precise normalisation in $\sqrt{t}$), [32] (for compact manifolds in the variable curvature case), [25] (for hyperbolic manifolds of finite volume) and more recently [7] (with a martingale argument which can be applied in weakly hyperbolic situations [24]). Let us emphasize that the above simple expression of the asymptotic variance σ_Φ^2 is obtained in a very elegant way in [7]; nevertheless, the proof given here simplifies Ratner's argument and does not require any speed of mixing of the geodesic flow, as in [7].

We can extend the Main Theorem in some others directions. First, one can fix a family $\{\omega_1, \ldots, \omega_\kappa\}$ of 1-forms with support on a neighborhood of $\mathcal{C}$ which represents a basis of $H^1(\mathcal{C}; \mathbb{R})$. Note that the homological rank κ of the parabolic group P may be smaller than its rank k if P is not abelian. We get therefore a multi-dimensional process $Y_t = (Y_t^1, \ldots, Y_t^\kappa)$ as above. This process has a limiting behavior and that the limit law is κ-dimensional stable, or Gaussian according to $\alpha < 2$ or $\alpha \geq 2$.

We shall also see that the processes X_t and Y_t becomes asymptotically independent when the cusp $\mathcal{C}$ is influential, i.e. $\alpha < 2$. An extension of this property is the following: one might consider all the cusps $\mathcal{C}_1, \ldots \mathcal{C}_p$ of the manifold simultaneously, and prove that the different processes built over the different cusps become asymptotically independent under m.