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KÄHLER MANIFOLDS WITH SPLIT TANGENT BUNDLE

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ABSTRACT. — This paper is concerned with compact Kähler manifolds whose tangent bundle splits as a sum of subbundles. In particular, it is shown that if the tangent bundle is a sum of line bundles, then the manifold is uniformised by a product of curves. The methods are taken from the theory of foliations of (co)dimension 1.

RÉSUMÉ (*Variétés kähleriennes à fibré tangent scindé*). — On étudie dans cet article les variétés kählériennes compactes dont le fibré tangent se décompose en somme directe de sous-fibrés. En particulier, on montre que si le fibré tangent se décompose en somme directe de sous-fibrés en droites, alors la variété est uniformisée par un produit de courbes. Les méthodes sont issues de la théorie des feuilletages de (co)dimension 1.

1. Introduction

We study in this paper compact Kähler manifolds whose tangent bundle splits as a sum of two or more subbundles. The basic result that we prove is the following theorem.

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THEOREM 1.1. — *Let M be a compact connected Kähler manifold. Suppose that its tangent bundle TM splits as $D \oplus L$, where $D \subset TM$ is a subbundle of codimension 1 and $L \subset TM$ is a subbundle of dimension 1. Then:*

(i) *If D is integrable then $\widetilde{M}$, the universal covering of M, splits as $\widetilde{N} \times E$, where E is a connected simply connected curve (the unit disc $\mathbb{D}$, the affine line $\mathbb{C}$ or the projective line $\mathbb{P}$). This splitting of $\widetilde{M}$ is compatible with the splitting of TM, in the sense that $T\widetilde{N} \subset T\widetilde{M}$ is the pull-back of D and $TE \subset T\widetilde{M}$ is the pull-back of L.*

(ii) *If D is not integrable then L is tangent to the fibres of a $\mathbb{P}$-bundle.*

This result will be the main ingredient in the proof of the following one. See also Section 4 for a more general statement.

THEOREM 1.2. — *Let M be a compact connected Kähler manifold whose tangent bundle splits as a sum of line subbundles:*

$$TM = L_1 \oplus \cdots \oplus L_n.$$

Then the universal covering $\widetilde{M}$ is isomorphic to a product of curves

$$\widetilde{M} = \mathbb{P}^r \times \mathbb{C}^s \times \mathbb{D}^t$$

for suitable integers r, s, t, $r + s + t = n$. Moreover, if all the codimension 1 subbundles $L_1 \oplus \cdots \oplus L_{j-1} \oplus L_{j+1} \oplus \cdots \oplus L_n$, $j = 1, \ldots, n$, are integrable, then the above splitting of $\widetilde{M}$ is compatible with the one of TM.

The problem of relating splitting properties of the tangent bundle of a compact complex manifold with splitting properties of the universal covering has been recently studied by Beauville [2], Druel [9], Campana-Peternell [7]. The point of view of these papers consists in analysing the interplay between splitting of the tangent bundle and some known differential-geometric or algebraic-geometric properties of the manifold. For instance, in [2] one makes use of Kähler-Einstein metrics, whereas in [9] and [7] a main tool is the geometry of rational curves on a projective variety (Mori theory).

Our point of view is completely independent on the geometry of the underlying manifold. On the contrary, it is completely dependent on the geometry of the foliations by curves generated by one dimensional subbundles of the tangent bundle. In some sense, we replace the Mori theory used in [7] with the "foliated" Mori theory funded by Miyaoka [3]. But also we like to work on compact Kähler manifolds which are possibly nonprojective, so that the algebraic point of view of [3] must be replaced by the more analytic one of [5] and [6], which moreover gives some useful metric-type information. Other simple but essential tools are the integrability criterion for codimension 1 distributions of [8] and the construction of holonomy invariant metrics for codimension 1 foliations of [4].

Roughly speaking, in the setting of Theorem 1.1 our method consists in constructing a special metric on the line bundle L. Then, in the setting of Theorem 1.2 and still roughly speaking, we shall obtain a special metric on M by summing the special metrics on the line bundles L_j, and this special metric on M will give the desired uniformisation. In this perspective, Theorem 1.2 should be compared with Simpson's uniformisation theorem [13, Cor. 9.7] (see also [2]), even if our construction of special metrics is completely different. In fact, we already have by free a special metric, given by the leafwise Poincaré metric, and we have just to verify that it is the good one.

2. One dimensional foliations with a transverse distribution

Let M be a compact connected Kähler manifold. Suppose that the tangent bundle TM splits as a sum of a one dimensional subbundle L and a codimension 1 subbundle D:

$$TM = D \oplus L.$$

The line subbundle L is tangent to a holomorphic one dimensional foliation $\mathcal{L}$. Each leaf of $\mathcal{L}$ is uniformized either by $\mathbb{P}$ (rational leaf) or by $\mathbb{C}$ (parabolic leaf) or by $\mathbb{D}$ (hyperbolic leaf). By a well known argument (Reeb stability plus compactness of the cycles space [11]), if some leaf is rational then every leaf is rational, and $\mathcal{L}$ is a locally trivial $\mathbb{P}$-bundle over some compact connected Kähler manifold N, $\dim N = \dim M - 1$.

In this case, the transverse distribution D may be integrable or not. If it is integrable, then foliation $\mathcal{D}$ generated by D can be described as a suspension of a representation of $\pi_1(N)$ into $\mathrm{A}ut(\mathbb{P})$, see [10, Ch. I]. It follows that $\widetilde{M}$, the universal covering of M, splits as $\widetilde{N} \times \mathbb{P}$, the splitting being compatible with the splitting of TM.

If D is not integrable and M is projective, a more subtle argument [7, §2] shows that $\widetilde{M}$ still splits as $\widetilde{N} \times \mathbb{P}$ (but now, of course, this splitting is no more compatible with $TM = D \oplus L$). Probably the same holds also in the Kähler nonprojective case, but we don't know a proof.

Let us now turn to the more interesting case in which no leaf is rational. We shall distinguish two different possibilities:

(a) There is a hyperbolic leaf;
(b) Every leaf is parabolic.

The following Proposition completes the proof of Theorem 1.1.

Proposition 2.1. — *In both cases* (a) *and* (b) *the distribution D is integrable, and generates a codimension* 1 *foliation $\mathcal{D}$. The holonomy of this foliation preserves a transverse hermitian metric of constant curvature κ, with $\kappa = -1$ in case* (a) *and $\kappa = 0$ in case* (b)*. The universal covering $\widetilde{M}$ splits as $\widetilde{N} \times E$, compatibly with $TM = D \oplus L$, and $E = \mathbb{D}$ in case* (a) *or $E = \mathbb{C}$ in case* (b).

2.1. The hyperbolic case. — If some leaf of $\mathcal{L}$ is hyperbolic, we shall rely on the main result of [5]: the leafwise Poincaré metric on $\mathcal{L}$ induces on $T^*\mathcal{L}$ ($= L^*$) a singular hermitian metric whose curvature is a closed positive current.

Let us fix an open covering $\{U_j\}$ of M, with holomorphic vector fields $v_j \in H^0(U_j, \Theta_M)$ generating $\mathcal{L}$ and holomorphic 1-forms (a priori, not necessarily integrable) $\omega_j \in H^0(U_j, \Omega^1_M)$ generating D. We may suppose, by the transversality condition, that $i_{v_j}\omega_j \equiv 1$. On overlapping charts we therefore have

$$v_i = g_{ij} v_j, \quad \omega_i = g_{ij}^{-1} \omega_j$$

where $g_{ij} : U_i \cap U_j \to \mathbb{C}^*$ are holomorphic functions forming a cocycle which defines the line bundle L^*.

For every j, set

$$h_j = \log \|v_j\|^2_{\mathrm{Poin}}$$

where $\|v_j(z)\|_{\mathrm{Poin}}$ is the norm of $v_j(z)$ with respect to the Poincaré metric on the leaf of $\mathcal{L}$ through z. The result of [5] recalled above says that h_j is a *plurisubharmonic* function. Recall also that, by definition, the Poincaré "metric" on a parabolic leaf is identically zero. Thus h_j may have poles, corresponding to the trace of parabolic leaves on U_j.

The arguments are very close to [4] and [8]. In fact, the integrability of D follows from [8] (L^* is the conormal bundle of D, and it is pseudoeffective), and the existence of a transverse metric invariant by the holonomy follows from [4, §§6–7]. But let us give anyway some detail for the sake of completeness and reader's convenience.

From $v_i = g_{ij} v_j$ we deduce that $h_i - h_j = \log |g_{ij}|^2$, and from $\omega_i = g_{ij}^{-1}\omega_j$ we see that the $(1,1)$-form locally defined by

$$\eta = \sqrt{-1}\,\mathrm{e}^{h_j} \omega_j \wedge \overline{\omega}_j$$

is indeed a well defined global positive $(1,1)$-form (with L^∞_{loc}-coefficients) on M. We may compute $\sqrt{-1}\partial\overline{\partial}\eta$, as a current. It turns out that it is a *positive* current.

Indeed, by the usual decomposition properties of positive forms, by $\eta \in L^\infty_{\mathrm{loc}}$, and by Fubini's theorem, it is sufficient to verify that for every local embedding $\iota : \mathbb{D}^2 \to M$ the current $\sqrt{-1}\partial\overline{\partial}(\iota^*\eta)$ is positive (that is, a positive measure on $\mathbb{D}^2$). If $\iota(\mathbb{D}^2)$ is tangent to D then $\iota^*\eta \equiv 0$. If $\iota(\mathbb{D}^2)$ is not tangent to D then the trace of D on $\iota(\mathbb{D}^2)$ defines a foliation outside a discrete subset $\Gamma \subset \mathbb{D}^2$. Thus, $\iota^*\omega_j$ outside Γ can be written, in suitable local coordinates (z, w), as $f\,\mathrm{d}z$, for some holomorphic function f. Consequently, $\iota^*\eta = \mathrm{e}^h |f|^2 \sqrt{-1}\,\mathrm{d}z \wedge \mathrm{d}\overline{z}$ and

$$\sqrt{-1}\partial\overline{\partial}(\iota^*\eta) = \sqrt{-1}\partial\overline{\partial}(\mathrm{e}^{h+\log|f|^2}) \wedge \sqrt{-1}\,\mathrm{d}z \wedge \mathrm{d}\overline{z}$$

which is positive because $h + \log |f|^2$ is plurisubharmonic.

This gives the positivity of $\sqrt{-1}\partial\overline{\partial}(\iota^*\eta)$ on $\mathbb{D}^2 \setminus \Gamma$. To obtain the positivity on the whole $\mathbb{D}^2$ we may simply use the extension theorem of [1]. The form $\iota^*\eta$

has bounded coefficients, so that if Θ is a Kähler form on $\mathbb{D}^2$ then $\iota^*\eta - c\Theta$ is a negative current for $c \gg 0$, whereas $\sqrt{-1}\partial\overline{\partial}(\iota^*\eta - c\Theta) = \sqrt{-1}\partial\overline{\partial}(\iota^*\eta)$ is positive outside Γ. By [1], this last one is positive on the full $\mathbb{D}^2$. Whence the positivity of $\sqrt{-1}\partial\overline{\partial}\eta$ on M.

By Stokes Theorem, the exact positive measure $\sqrt{-1}\partial\overline{\partial}\eta\wedge\Theta^{n-2}$ (Θ is now a Kähler form on M, and $n = \dim M$) must be identically zero, so that $\sqrt{-1}\partial\overline{\partial}\eta$ is also identically zero:

$$\sqrt{-1}\partial\overline{\partial}\eta \equiv 0.$$

Looking again at the local restriction $\iota^*\eta$, $\iota : \mathbb{D}^2 \to M$, we obtain that the function $\mathrm{e}^{h+\log|f|^2}$ is harmonic in the w-variable. Because $h + \log|f|^2$ is w-subharmonic, the only possibility is that $h + \log|f|^2$ is w-constant: the exponential of a nonconstant subharmonic function is strictly subharmonic. This implies that $\iota^*\eta$ is not only $\partial\overline{\partial}$-closed, but also d-closed. By varying the embedding $\iota : \mathbb{D}^2 \to M$, we obtain:

$$\mathrm{d}\eta \equiv 0.$$

This means two things:

(i) the distribution $D = \ker\eta$ is integrable, and hence generates a codimension 1 foliation $\mathcal{D}$;

(ii) on the transversals to $\mathcal{D}$, η induces a measure invariant by the holonomy.

Remark that all of this uses only the fact that L^*, the conormal bundle of D, is pseudoeffective, i.e., the functions h_j are plurisubharmonic. But, by the normalisation $i_{v_j}\omega_j \equiv 1$ and the definition of h_j, we see that the restriction of η to the leaves of $\mathcal{L}$ is nothing but than the area form of the hyperbolic metric on those leaves. Therefore, the holonomy of $\mathcal{D}$ preserves that hyperbolic metric.

In order to complete the proof of Proposition 2.1, case (a), it remains only to prove the splitting property of $\widetilde{M}$. This will follow from a general Splitting Lemma which we postpone to Section 3.

2.2. The parabolic case. — If all the leaves of $\mathcal{L}$ are parabolic, the leafwise Poincaré metric is identically zero and we cannot say, a priori, that L^* is pseudoeffective (unless M is projective, by [3]). But we shall see that indeed it is, and it is even flat, thanks to the existence of the transverse distribution D.

The starting point is the following one [6]: if $T \subset M$ is a codimension 1 disc transverse to $\mathcal{L}$, then the associated covering tube U_T (union of the universal coverings of the leaves through T) is holomorphically trivial: $U_T \simeq T \times \mathbb{C}$. This fact can be reformulated in the following way. Take a foliated chart $U \simeq T\times\mathbb{D} \subset M$ around $T = T\times\{0\}$. Then any nonvanishing section v_0 of $T\mathcal{L}|_T$ (i.e. a vector field tangent to $\mathcal{L}$ at points of T) can be extended to a section v of $T\mathcal{L}|_U$ in a canonical way: for every $t \in T$, we simply require that $v|_{\{t\}\times\mathbb{D}}$ is the restriction to the plaque $\{t\} \times \mathbb{D}$ of a *complete nonsingular* vector field on the leaf of $\mathcal{L}$ containing $\{t\} \times \mathbb{D}$. This is well defined, because on a parabolic