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ON COVERINGS OF SIMPLE ABELIAN VARIETIES

by Olivier Debarre

Abstract. — To any finite covering $f : Y \to X$ of degree d between smooth complex projective manifolds, one associates a vector bundle E_f of rank $d-1$ on X whose total space contains Y. It is known that E_f is ample when X is a projective space ([9]), a Grassmannian ([11]), or a Lagrangian Grassmannian ([7]). We show an analogous result when X is a simple abelian variety and f does not factor through any nontrivial isogeny $X' \to X$. This result is obtained by showing that E_f is M-regular in the sense of Pareschi-Popa, and that any M-regular sheaf is ample.

Résumé (*Sur les revêtements des variétés abéliennes simples*). — On associe à tout revêtement fini $f : Y \to X$ de degré d entre variétés projectives lisses complexes un fibré vectoriel E_f de rang $d-1$ sur X dont l'espace total contient Y. On sait que E_f est ample lorsque X est un espace projectif ([9]), une grassmannienne ([11]) ou une grassmannienne lagrangienne ([7]). Nous montrons un résultat analogue lorsque X est une variété abélienne simple et que f ne se factorise par aucune isogénie non triviale $X' \to X$. Ce résultat est obtenu en montrant que E_f est M-régulier au sens de Pareschi-Popa, puis que tout faisceau M-régulier est ample.

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1. Introduction

We work over the complex numbers. Let $f : Y \to X$ be a finite surjective morphism of degree d between smooth projective varieties of the same dimension n. The morphism f is flat, hence the sheaf $f_*\mathcal{O}_Y$ is locally free. We may define a locally free sheaf E_f of rank $d-1$ on X as the dual of the kernel of the trace map $\mathrm{Tr}_{Y/X} : f_*\mathcal{O}_Y \to \mathcal{O}_X$, so that

$$f_*\mathcal{O}_Y = \mathcal{O}_X \oplus E_f^*$$

By duality for a finite flat morphism, we have

$$f_*\omega_{Y/X} = \mathcal{O}_X \oplus E_f$$

Our aim is to prove the following statement conjectured in [1].

THEOREM 1.1. — *Let X be a* simple *abelian variety, let Y be a smooth* connected *projective variety, and let $f : Y \to X$ be a finite cover. If f does not factor through any nontrivial isogeny $X' \to X$, the vector bundle E_f is ample.*

For a more general statement, see Theorem 4.1. See also the remarks at the end of this article for more comments. Even if X is not simple, the vector bundle E_f is known to be nef (see [14, Theorem 1.17], [10, Example 6.3.59]) and its restriction to a general complete intersection curve in X to be ample (see [6, Lemma 2.7]).

The ampleness of E_f has a number of consequences, as explained in [10, Example 6.3.56]. In our case, one new statement beyond the Fulton-Hansen-type results already obtained in [1] is the following: under the hypotheses of the theorem, the induced morphism

$$H^i(f, \mathbb{C}) : H^i(X, \mathbb{C}) \longrightarrow H^i(Y, \mathbb{C})$$

is bijective for $i \leq n - d + 1$ (see [10, Theorem 7.1.16]).

When moreover $d \leq n$, the morphism $\pi_1(f) : \pi_1(Y) \to \pi_1(X)$ is bijective.(1) In particular, the group $H_1(Y, \mathbb{Z})$ is isomorphic to $H_1(X, \mathbb{Z})$, hence is torsion-free, and so is $H^2(Y, \mathbb{Z})$ by the universal coefficient theorem.

When $d \leq n-1$, the morphism $H^2(f, \mathbb{Z}) : H^2(X, \mathbb{Z}) \to H^2(Y, \mathbb{Z})$ is injective with finite cokernel, hence so is $\mathrm{Pic}(f) : \mathrm{Pic}(X) \to \mathrm{Pic}(Y)$. It seems likely that those two maps are bijective.

The proof is a simple application of the results of [13] about global generation of sheaves on an abelian variety. More precisely, it is based on the remark that any M-regular sheaf (§ 3) on an abelian variety is ample (Corollary 3.2).

(1) For algebraic fundamental groups, this is [1, Corollaire 6.2]; for topological fundamental groups, this is [2, Exercice VIII.5], where the hypothesis $d \leq n$ is unfortunately missing.

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2. Ample sheaves

To any coherent sheaf $\mathcal{F}$ on a scheme X of finite type over $\mathbb{C}$, one associates the X-scheme

$$\mathbf{P}(\mathcal{F}) = \operatorname{Proj}\Big(\bigoplus_{m\geq 0} \mathbf{Sym}^m\mathcal{F}\Big)$$

and an invertible sheaf $\mathcal{O}_{\mathbf{P}(\mathcal{F})}(1)$ on $\mathbf{P}(\mathcal{F})$. The sheaf $\mathcal{F}$ is said to be ample if $\mathcal{O}_{\mathbf{P}(\mathcal{F})}(1)$ is.

Well-known properties of ampleness for locally free sheaves (see for example [10, Chapter 6]) still hold in this general setting:

a) the sheaf $\mathcal{F}$ is ample if and only if, for any coherent sheaf $\mathcal{G}$ on X, the sheaf $\mathcal{G}\otimes\mathbf{Sym}^m\mathcal{F}$ is globally generated for all $m\gg 0$ (see [8, Theorem 1]);

b) any quotient of an ample sheaf is ample (see [8, Proposition 1]);

c) if $\pi : Y\to X$ is a finite morphism, $\mathcal{F}$ is ample if and only if $\pi^*\mathcal{F}$ is (this is because $\mathbf{P}(\pi^*\mathcal{F}) = \mathbf{P}(\mathcal{F})\times_X Y$ and $\mathcal{O}_{\mathbf{P}(\mathcal{F})}(1)$ pulls back, by a finite morphism, to $\mathcal{O}_{\mathbf{P}(\pi^*\mathcal{F})}(1)$);

d) if X is proper and $\mathcal{F}$ is globally generated, $\mathcal{F}$ is ample if and only if, for any curve C in X, the restriction $\mathcal{F}\otimes\mathcal{O}_C$ has no trivial quotient (Gieseker's Lemma).

3. Continuously generated sheaves

Following [13, Definition 2.10], we say that a coherent sheaf $\mathcal{F}$ on an irreducible projective variety X is *continuously globally generated* if, for any nonempty subset U of $\operatorname{Pic}^0(X)$, the sum of the twisted evaluation maps

$$\bigoplus_{\xi\in U} H^0(X,\mathcal{F}\otimes P_\xi)\otimes P_\xi^\vee \longrightarrow \mathcal{F}$$

is surjective, where, for any element ξ of $\operatorname{Pic}^0(X)$, we denote by P_ξ the corresponding numerically trivial line bundle on X. This property is equivalent to the existence of a positive integer N such that for $(\xi_1,\dots,\xi_N)$ general in $\operatorname{Pic}^0(X)^N$, the analogous map

$$\bigoplus_{i=1}^{N} H^0(X,\mathcal{F}\otimes P_{\xi_i})\otimes P_{\xi_i}^\vee \longrightarrow \mathcal{F} \tag{1}$$

is surjective. Being a quotient of a direct sum of numerically trivial line bundles, a continuously globally generated sheaf is nef. Our aim is to show that under certain circumstances, it is ample.

Proposition 3.1. — *A coherent sheaf $\mathcal{F}$ on an irreducible projective variety X is continuously globally generated if and only if there exists a connected abelian Galois étale cover $\pi : Y \to X$ such that $\pi^*(\mathcal{F}\otimes P_\xi)$ is globally generated for all $\xi \in \operatorname{Pic}^0(X)$.*

Proof. — Assume $\mathcal{F}$ is continuously globally generated and let $\xi \in \operatorname{Pic}^0(X)$. Since torsion points are dense in $\operatorname{Pic}^0(X)^N$, the open subset of $\operatorname{Pic}^0(X)^N$ of points for which the map (1) is surjective and all $h^0(X, \mathcal{F}\otimes P_{\xi_i})$ are minimal contains a point of the type

$$\bigl(\xi+\eta_1(\xi),\dots,\xi+\eta_N(\xi)\bigr)$$

where $(\eta_1(\xi),\dots,\eta_N(\xi))$ is torsion, hence contains also $U_\xi + (\eta_1(\xi),\dots,\eta_N(\xi))$, where U_ξ is a neighborhood of ξ in $\operatorname{Pic}^0(X)$. Since $\operatorname{Pic}^0(X)$ is quasi-compact, it is covered by finitely many such neighborhoods, say $U_{\xi_1},\dots,U_{\xi_M}$.

Let $\pi : Y \to X$ be a connected abelian Galois étale cover such that the kernel of $\operatorname{Pic}^0(\pi) : \operatorname{Pic}^0(X) \to \operatorname{Pic}^0(Y)$ contains all $\eta_i(\xi_j)$, for $i \in \{1,\dots,N\}$ and $j\in\{1,\dots,M\}$. Fix $j \in \{1,\dots,M\}$; the map

$$\bigoplus_{i=1}^N H^0(X, \mathcal{F}\otimes P_\xi\otimes P_{\eta_i(\xi_j)})\otimes \pi^*P_\xi^\vee \otimes \pi^* P^\vee_{\eta_i(\xi_j)} \longrightarrow \pi^*\mathcal{F}$$

is surjective for all $\xi \in U_{\xi_j}$. But this map is

$$\bigoplus_{i=1}^N H^0(X, \mathcal{F}\otimes P_\xi\otimes P_{\eta_i(\xi_j)})\otimes \pi^*P_\xi^\vee \longrightarrow \pi^*\mathcal{F}$$

and since each $H^0(X,\mathcal{F}\otimes P_\xi \otimes P_{\eta_i(\xi_j)})$ is a vector subspace of $H^0(Y,\pi^*(\mathcal{F}\otimes P_\xi))$, the sheaf $\pi^*(\mathcal{F}\otimes P_\xi)$ is globally generated for all $\xi \in U_{\xi_j}$, hence for all ξ in $\operatorname{Pic}^0(X)$.

For the converse, assume that there exists a connected abelian Galois étale cover $\pi : Y\to X$ such that the evaluation map

$$H^0\bigl(Y, \pi^*(\mathcal{F}\otimes P_\xi)\bigr)\otimes \mathcal{O}_Y \longrightarrow \pi^*(\mathcal{F}\otimes P_\xi)$$

is surjective for all $\xi\in\operatorname{Pic}^0(X)$. Since π is finite, the map

$$H^0(X,\mathcal{F}\otimes P_\xi\otimes \pi_*\mathcal{O}_Y)\otimes \pi_*\mathcal{O}_Y \longrightarrow \mathcal{F}\otimes P_\xi \otimes \pi_*\mathcal{O}_Y$$

is also surjective. If we let $\operatorname{Ker}(\operatorname{Pic}^0(\pi)) = \{\eta_1,\dots,\eta_N\}$, we have $\pi_*\mathcal{O}_Y = \bigoplus_{i=1}^N P_{\eta_i}$, the map

$$\Bigl(\bigoplus_{i=1}^N H^0(X,\mathcal{F}\otimes P_\xi\otimes P_{\eta_i})\Bigr)\otimes\Bigl(\bigoplus_{i=1}^N P_{\eta_i}\Bigr) \longrightarrow \mathcal{F}\otimes P_\xi\otimes\Bigl(\bigoplus_{i=1}^N P_{\eta_i}\Bigr)$$

is surjective, and so is

$$\bigoplus_{i=1}^{N} H^0(X, \mathcal{F} \otimes P_\xi \otimes P_{\eta_i}) \otimes P_{\eta_i}^\vee \longrightarrow \mathcal{F} \otimes P_\xi.$$

In other words, the map (1) is surjective for $(\xi_1, \ldots, \xi_N) = (\xi + \eta_1, \ldots, \xi + \eta_N)$, for all $\xi \in \operatorname{Pic}^0(X)$. Choosing ξ_0 such that $h^0(X, \mathcal{F} \otimes P_{\xi_0+\eta_i})$ takes the general (minimal) value for each i in $\{1, \ldots, N\}$, we obtain that the map (1) is still surjective for $(\xi_1, \ldots, \xi_N)$ in a neighborhood of $(\xi_0 + \eta_1, \ldots, \xi_0 + \eta_N)$. This proves that $\mathcal{F}$ is continuously globally generated. ☐

Corollary 3.2. — *Let X an irreducible projective variety with a finite map to an abelian variety. Any continuously globally generated coherent sheaf on X is ample.*

Proof. — Let $\mathcal{F}$ be a continuously globally generated coherent sheaf on X. By Proposition 3.1, there exists a connected abelian Galois étale cover $\pi : Y \to X$ such that $\pi^*(\mathcal{F} \otimes P_\xi)$ is globally generated for all $\xi \in \operatorname{Pic}^0(X)$.

Let C be a curve in Y. If there is a trivial quotient $\pi^*\mathcal{F}_{|C} \twoheadrightarrow \mathcal{O}_C$, we have also surjections $\pi^*(\mathcal{F} \otimes P_\xi)_{|C} \twoheadrightarrow \pi^* P_{\xi|C}$ for each $\xi \in \operatorname{Pic}^0(X)$. Since $\pi^*(\mathcal{F} \otimes P_\xi)$ is globally generated, so is $\pi^* P_{\xi|C}$. This implies that the composition $\operatorname{Pic}^0(X) \to \operatorname{Pic}^0(Y) \to \operatorname{Pic}^0(C)$ is zero, hence that $\pi(C)$ is contracted by any map from X to an abelian variety. This contradicts our hypothesis, hence $\pi^*\mathcal{F}_{|C}$ has no trivial quotient.

By Gieseker's Lemma, $\pi^*\mathcal{F}$ is ample, and so is $\mathcal{F}$ (§ 2). ☐

4. The main theorem

Following [13, Definition 2.1], we say that a coherent sheaf $\mathcal{F}$ on an abelian variety A is *M-regular* if

$$\operatorname{codim}_{\operatorname{Pic}^0(A)} \operatorname{Supp}\big(R^i\widehat{\mathcal{S}}(\mathcal{F})\big) > i$$

for all $i > 0$ ($R^i\widehat{\mathcal{S}}$ is the ith Fourier-Mukai functor). This is the case if

$$\operatorname{codim}_{\operatorname{Pic}^0(A)}\{\xi \in \operatorname{Pic}^0(A) \mid H^i(A, \mathcal{F} \otimes P_\xi) \neq 0\} > i$$

for all $i > 0$. We refer to [12] and [13] for more details. For our purposes, the main result of [13] (Proposition 2.13) is that *an M-regular coherent sheaf on an abelian variety is continuously globally generated.*

Theorem 4.1. — *Let X be a smooth connected projective variety with a finite map to a* simple *abelian variety, let Y be a smooth* connected *projective variety with a finite surjective map $f : Y \to X$. If f factors through no nontrivial connected abelian Galois étale covering of X, the vector bundle $E_f \otimes \omega_X$ is ample.*