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IRREGULARITY OF AN ANALOGUE OF THE GAUSS-MANIN SYSTEMS

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ABSTRACT. — In $\mathcal{D}$-modules theory, Gauss-Manin systems are defined by the direct image of the structure sheaf $\mathcal{O}$ by a morphism. A major theorem says that these systems have only regular singularities. This paper examines the irregularity of an analogue of the Gauss-Manin systems. It consists in the direct image complex $f_+(\mathcal{O}\mathrm{e}^g)$ of a $\mathcal{D}$-module twisted by the exponential of a polynomial g by another polynomial f, where f and g are two polynomials in two variables. The analogue of the Gauss-Manin systems can have irregular singularities (at finite distance and at infinity). We express an invariant associated with the irregularity of these systems at $c \in \mathbb{P}^1$ by the geometry of the map (f, g).

RÉSUMÉ (*Irrégularité d'un analogue des systèmes de Gauss-Manin*)

Dans la théorie des $\mathcal{D}$-modules, on définit les systèmes de Gauss-Manin par l'image directe par un morphisme du faisceau structural $\mathcal{O}$. Un résultat essentiel est leur régularité. On s'intéresse à l'irrégularité d'un analogue des systèmes de Gauss-Manin. Il s'agit de l'image directe $f_+(\mathcal{O}\mathrm{e}^g)$ par un polyôme f d'un $\mathcal{D}$-module tordu par une exponentielle d'un second polynôme g, où f et g sont des polynômes à deux variables. Les analogues des systèmes de Gauss-Manin peuvent avoir des singularités irrégulières. On exprimera alors un invariant attaché à l'irrégularité en $c \in \mathbb{P}^1$ de ces systèmes à l'aide de la géométrie de l'application (f, g).

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Contents

1. Introduction

1.1. We denote by $\mathcal{O}_{\mathbb{C}^n}$ the sheaf of regular functions on $\mathbb{C}^n$ and by $\mathcal{D}_{\mathbb{C}^n}$ the sheaf of algebraic differential operators on $\mathbb{C}^n$.

Let $f:\mathbb{C}^n\to\mathbb{C}$ be a polynomial. In $\mathcal{D}$-module theory, we define the Gauss-Manin systems as the cohomology modules of a complex of $\mathcal{D}_{\mathbb{C}}$-modules, it being the direct image complex $f_+(\mathcal{O}_{\mathbb{C}^n})$. They are holonomic and regular. These systems coincide with the Gauss-Manin connections outside a finite subset Σ of $\mathbb{C}$ such that $f:f^{-1}(\mathbb{C}\setminus\Sigma)\to\mathbb{C}\setminus\Sigma$ is a locally trivial fibration.

Now, let $g:\mathbb{C}^n\to\mathbb{C}$ be another polynomial. We denote by $\mathcal{O}_{\mathbb{C}^n}\,\mathrm{e}^g$ the $\mathcal{D}_{\mathbb{C}^n}$-module obtained from $\mathcal{O}_{\mathbb{C}^n}$ by twisting by e^g. We are interested in an analogue of the Gauss-Manin systems, it being the direct image complex $f_+(\mathcal{O}_{\mathbb{C}^n}\,\mathrm{e}^g)$.

In [7], F. Maaref calculates the generic fibre of the sheaf of horizontal analytic sections of the systems $\mathcal{H}^k(f_+(\mathcal{O}_{\mathbb{C}^n}\,\mathrm{e}^g))$. It consists in a relative version of a result of C. Sabbah in [12]. Indeed, the generic fiber of the sheaf of horizontal analytic sections of $\mathcal{H}^k(f_+(\mathcal{O}_{\mathbb{C}^n}\,\mathrm{e}^g))$ is canonically isomorphic to the cohomology group with closed support $H^{k+n-1}_{\Phi_t}(f^{-1}(t)^{\mathrm{an}},\mathbb{C})$, where Φ_t is a family of closed subsets of $f^{-1}(t)$, on which e^{-g} is rapidly decreasing. More precisely, this family is defined as follow. Let $\pi:\widetilde{\mathbb{P}^1}\to\mathbb{P}^1$ be the oriented real blow-up of $\mathbb{P}^1$ at infinity. $\widetilde{\mathbb{P}^1}$ is diffeomorphic to $\mathbb{C}\cup S^1$, where S^1 is the circle of directions at infinity. A is in Φ_t if A is a closed subset of $f^{-1}(t)$ and the closure of $g(A)$ in $\mathbb{C}\cup S^1$ intersects S^1 in $]-\frac{1}{2}\pi,\frac{1}{2}\pi[$.

This isomorphism can be better understood using relative cohomology group. F. Maaref shows that for all $t\notin\Sigma$ and for all ρ, such that $\mathrm{Re}(-\rho)$ is sufficiently large, the fibre at t of the sheaf of horizontal analytic sections of $\mathcal{H}^k(f_+(\mathcal{O}_{\mathbb{C}^n}\,\mathrm{e}^g))$ is isomorphic to the relative cohomology group $H^{k+n-1}(f^{-1}(t)^{\mathrm{an}},(f^{-1}(t)\cap g^{-1}(\rho))^{\mathrm{an}},\mathbb{C})$.

Finally, he proves the quasi-unipotence of the corresponding local monodromy.

1.2. The Gauss-Manin systems have only regular singularities. In our case, the complex $f_+(\mathcal{O}_{\mathbb{C}^n}\,\mathrm{e}^g)$ can have irregular singularities. The aim of this paper is to characterize this irregularity in terms of the geometry of the map (f, g), when f and g are two polynomials in two variables.

f and g are algebraically independant, we will prove that the complex $f_+(\mathcal{O}_{\mathbb{C}^2}\,\mathrm{e}^g)$ is essentially concentrated in degree zero. Then, we can associate to this complex a system of differential equations in one variable. We want to calculate the irregularity number of this system at a point at finite distance and at infinity.

Let $\mathbb{X}$ be a smooth projective compactification of $\mathbb{C}^2$ such that there exists $F, G : \mathbb{X} \to \mathbb{P}^1$, two meromorphic maps, which extend f and g. Let us denote by D the divisor $\mathbb{X} \setminus \mathbb{C}^2$. In the following, we identify $\mathbb{P}^1$ with $\mathbb{C} \cup \{\infty\}$.

Let Γ be the critical locus of (F, G). We denote by Δ_1 the cycle in $\mathbb{P}^1 \times \mathbb{P}^1$ which is the closure in $\mathbb{P}^1 \times \mathbb{P}^1$ of $(F, G)(\Gamma) \cap (\mathbb{C}^2 \setminus \{c\} \times \mathbb{C})$ where the image is counted with multiplicity and by Δ_2 the cycle in $\mathbb{P}^1 \times \mathbb{P}^1$ which is the closure in $\mathbb{P}^1 \times \mathbb{P}^1$ of $(F, G)(D) \cap (\mathbb{C}^2 \setminus \{c\} \times \mathbb{C})$ where the image is counted with multiplicity.

For all $c \in \mathbb{P}^1$, the germs at (c, ∞) of the support of Δ_1 and Δ_2 are some germs of curves or are empty. Then, we denote by $I_{(c,\infty)}(\Delta_i, \mathbb{P}^1 \times \{\infty\})$ the intersection number of the cycles Δ_i and $\mathbb{P}^1 \times \{\infty\}$. If the germ at (c, ∞) of Δ_i is empty, this number is equal to 0.

Theorem 1. — *Let $f, g \in \mathbb{C}[x, y]$ be algebraically independant. Let $c \in \mathbb{P}^1$. Then, the irregularity number at c of the system $\mathcal{H}^0(f_+(\mathcal{O}_{\mathbb{C}^2}\,\mathrm{e}^g))$ is equal to*

$$I_{(c,\infty)}\big(\Delta_1, \mathbb{P}^1 \times \{\infty\}\big) + I_{(c,\infty)}\big(\Delta_2, \mathbb{P}^1 \times \{\infty\}\big).$$

When $c \in \mathbb{C}$, we can prove that the germ at (c, ∞) of Δ_2 is empty. Moreover, the germ at (c, ∞) of Δ_1 coincide with the one of the closure in $\mathbb{P}^1 \times \mathbb{P}^1$ of $(f, g)(\widetilde{\Gamma}) \setminus \{c\} \times \mathbb{C}$, where $\widetilde{\Gamma}$ is the critical locus of (f, g).

1.3. In general, we do not know how to calculate directly the irregularity number of a system associated with $f_+(\mathcal{O}_{\mathbb{C}^2}\,\mathrm{e}^g)$. The notion of irregularity complex along an hypersurface defined by Z. Mebkhout (see [9] and [10]) is the appropriate tool to express the irregularity of $f_+(\mathcal{O}_{\mathbb{C}^2}\,\mathrm{e}^g)$ (see §2). Indeed, this irregularity complex along an hypersurface is a generalization of the irregularity number at a point of a system of differential equations in one variable. Moreover, Z. Mebkhout proves a theorem of commutation between the direct image functor and the irregularity functor (see Theorem 2.4). Then, the irregularity number at $c \in \mathbb{P}^1$ of the system of differential equations associated with $f_+(\mathcal{O}_{\mathbb{C}^2}\,\mathrm{e}^g)$ can be expressed with the help of an irregularity complex of a $\mathcal{D}$-module in two variables along a curve.

In the general case where f and g are not necessarily algebraically independant, the complex $f_+(\mathcal{O}_{\mathbb{C}^2}\,\mathrm{e}^g)$ is not necessarily concentrated in degree 0. Then, we want to calculate the alternative sum of the irregularity number at $c \in \mathbb{P}^1$ of the systems $\mathcal{H}^k(f_+(\mathcal{O}_{\mathbb{C}^2}\,\mathrm{e}^g))$. This irregularity number IR_c is equal to the Euler characteristic of a complex of vector spaces over $\mathbb{C}$, it being the irregularity complex of $f_+(\mathcal{O}_{\mathbb{C}^2}\mathrm{e}^g)$ at $c \in \mathbb{P}^1$. When f and g are algebraically independant, this number coincide with the irregularity number of the system $\mathcal{H}^0(f_+(\mathcal{O}_{\mathbb{C}^2}\mathrm{e}^g))$. Then, we can prove that the irregularity number IR_c is equal to

$$-\chi\big(\mathbb{R}\Gamma\big(F^{-1}(c)\cap G^{-1}(\infty), \mathrm{IR}_{F^{-1}(c)}(\mathcal{O}_{\mathbb{X}}[*D]\mathrm{e}^G)\big)\big),$$

where $\mathrm{IR}_{F^{-1}(c)}(\mathcal{O}_{\mathbb{X}}[*D]\mathrm{e}^G)$ is the irregularity complex of $\mathcal{O}_{\mathbb{X}}[*D]\mathrm{e}^G$ along $F^{-1}(c)$.

Then, according to C. Sabbah [12], we know that, for $x \in F^{-1}(c)\cap G^{-1}(\infty)$, the Euler characteristic of $(\mathrm{IR}_{F^{-1}(c)}(\mathcal{O}_{\mathbb{X}}[*D]\mathrm{e}^G))_x$ is equal to the Euler characteristic of the fiber $f^{-1}(D^*(c,\eta))\cap g^{-1}(\rho)\cap B(x,\epsilon)$, where ϵ and η are small enough and $|\rho|$ is big enough. This result is stated in Theorem 3.4, §3 in terms of complex of nearby cycles.

Then, we have to globalize the situation (see §4). First of all, we prove that for η small enough and R big enough,

$$g : f^{-1}\big(D^*(c,\eta)\big)\cap g^{-1}\big(\{|\rho| > R\}\big) \longrightarrow \big\{|\rho| > R\big\}$$

is a locally trivial fibration. Then, the irregularity number IR_c is equal to the opposite of the Euler characteristic of its fiber $f^{-1}(D^*(c,\eta))\cap g^{-1}(\rho)$. This result hold in the general case where f and g are not necessarily algebraically independant.

Then, we have to study the topology of this fiber. We have to distinguished the case where f and g are algebraically independant (see §5) and the one where they are algebraically dependant (see §6).

2. Irregularity complex along an hypersurface

We will use the definition of regularity given by Z. Mebkhout [9], [10]. First of all, we recall the definition of irregularity complex of analytic $\mathcal{D}$-modules. Then, we define the notion of irregularity complex for algebraic $\mathcal{D}$-modules. Here, we have to take into account the behaviour of these modules at infinity. Moreover, we state major theorems on irregularity: the positivity theorem, the stability of the category of complex of regular holonomic $\mathcal{D}$-modules (analytic) by direct image by a proper map and the comparison theorem of Grothendieck.

2.1. The analytic case. — Let X be a smooth analytic variety over $\mathbb{C}$. In this section, $\mathcal{D}_X$ denotes the sheaf of analytic differential operators on X.

Let Z be an analytic closed subset of X. Denote by i the canonical inclusion of $X \setminus Z$ in X. Let $\mathcal{M}^\bullet$ be a bounded complex of analytic $\mathcal{D}_X$-modules with holonomic cohomology.

DEFINITION 2.1. — We define the *irregularity complex* of $\mathcal{M}^\bullet$ along Z as the complex

$$\begin{aligned}\mathrm{IR}_Z(\mathcal{M}^\bullet) &:= R\Gamma_Z\big(DR(\mathcal{M}^\bullet[*Z])\big)[+1]\\ &:= \mathrm{cone}\big(DR(\mathcal{M}[*Z]) \to Ri_*i^{-1}(DR(\mathcal{M}^\bullet[*Z]))\big).\end{aligned}$$

According to the constructibility theorem (cf. [6] and [11]), this complex is a bounded complex of constructible sheaves on X with support in Z. Then, we can define the covariant exact functor IR_Z between the category of bounded complexes of $\mathcal{D}_X$-modules with holonomic cohomology and the category of bounded complexes of constructible sheaves on X with support in Z.

DEFINITION 2.2. — $\mathcal{M}^\bullet$ is said to be *regular* if its irregularity complex along all hypersurfaces of X is zero.

In one variable, the previous definition of regularity generalises the notion of regular singular point of a differential equation which is characterized by the annulation of the irregularity number (Fuchs Theorem). Indeed, irregularity complex along an hypersurface generalizes irregularity number in the case of one variable. According to Z. Mebkhout [9], [10], the characteristic cycle of the irregularity complex of a holonomic $\mathcal{D}$-module along an hypersurface is positive.

THEOREM 2.3 (Positivity Theorem). — *If Z is an hypersurface of X and $\mathcal{M}$ is a holonomic $\mathcal{D}_X$-module, the complex $\mathrm{IR}_Z(\mathcal{M})$ is perverse on Z.*

The category of complexes of $\mathcal{D}$-modules with regular holonomic cohomology is stable by proper direct image. Let us state the theorem which proves this stability (see [9] and Proposition 3.6-4 of [10]). It will be a major tool in this paper.

Let $\pi : X \to Y$ be a proper morphism of smooth analytic varieties over $\mathbb{C}$. Let T be a hypersurface of Y.

THEOREM 2.4. — *Let $\mathcal{M}^\bullet$ be a bounded complex of analytic $\mathcal{D}_X$-modules with holonomic cohomology. We have an isomorphism*

$$\mathrm{IR}_T\big(\pi_+(\mathcal{M}^\bullet)\big)[\dim Y] \simeq R\pi_*\big(\mathrm{IR}_{\pi^{-1}(T)}(\mathcal{M}^\bullet)\big)[\dim X].$$

2.2. The algebraic case. — Let X be a smooth affine variety over $\mathbb{C}$. In this section, $\mathcal{D}_X$ denotes the sheaf of algebraic differential operators on X. Denote by $j : X \to \mathbb{P}^n$ an immersion of X in a projective space. Let Z be a locally closed subvariety of $\mathbb{P}^n$ and $\mathcal{M}^\bullet$ a bounded complex of algebraic $\mathcal{D}_X$-modules with holonomic cohomology.

DEFINITION 2.5. — We define the *irregularity complex* of $\mathcal{M}^\bullet$ along Z as

$$\mathrm{IR}_Z(j_+(\mathcal{M}^\bullet)) := \mathrm{IR}_{Z^{\mathrm{an}}}(j_+(\mathcal{M}^\bullet)^{\mathrm{an}}),$$