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CASCADE OF PHASES IN TURBULENT FLOWS

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ABSTRACT. — This article is devoted to incompressible Euler equations (or to Navier-Stokes equations in the vanishing viscosity limit). It describes the propagation of *quasi-singularities*. The underlying phenomena are consistent with the notion of a *cascade of energy*.

RÉSUMÉ (*Cascade de phases pour des fluides turbulents*). — Cet article étudie les équations d'Euler incompressible (ou de Navier-Stokes en présence de viscosité évanescente). On y décrit la propagation de *quasi-singularités*. Les phénomènes sous-jacents confirment l'idée selon laquelle il se produit une *cascade d'énergie*.

1. Introduction

Consider incompressible fluid equations

$$(\mathcal{E}) \qquad \partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla \boldsymbol{p} = 0, \quad \operatorname{div} \boldsymbol{u} = 0, \quad (t,x) \in [0,T] \times \mathbb{R}^d,$$

where $\boldsymbol{u} = {}^t(\boldsymbol{u}^1, \dots, \boldsymbol{u}^d) \in \mathbb{R}^d$ is the fluid velocity and $\boldsymbol{p} \in \mathbb{R}$ is the pressure. The structure of *weak* solutions of $(\mathcal{E})$ in d-space dimensions with $d \geq 2$ is a problem of wide current interest [3], [5], [25]. The questions are how to describe

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the phenomena with adequate models and how to visualize the results in spite of their complexity. We will achieve a small step in these two directions.

According to the physical intuition, the appearance of singularities is linked with the *increase of the vorticity*. Along this line, we have to mark the contributions [2] and [10]. Interesting objects are solutions which do not blow up in finite time but whose associated vorticities increase arbitrarily fast. These are *quasi-singularities*. Their study is of practical importance.

Typical examples of quasi-singularities are oscillations. This is a well-known fact going back to [4], [26]. The works [4] and [26] rely on phenomenological considerations and engineering experiments. Further developments are related to homogenization [14], [15], compensated compactness [12], [18] and non linear geometric optics [7], [8], [9].

DiPerna and Majda [12] show the persistence of oscillations in three dimensional Euler equations ($d = 3$). To this end, they select parameters $\varepsilon \in]0,1]$ and look at

$$\boldsymbol{u}_s^\varepsilon(t,x) := {}^t\big(\boldsymbol{g}(x_2,\varepsilon^{-1}x_2), 0, \boldsymbol{h}(x_1 - \boldsymbol{g}(x_2,\varepsilon^{-1}x_2)t, x_2, \varepsilon^{-1}x_2)\big) \tag{1.1}$$

where $\boldsymbol{g}(x_2,\theta)$ and $\boldsymbol{h}(x_1,x_2,\theta)$ are smooth bounded functions with period 1 in θ. They remark that the functions $\boldsymbol{u}_s^\varepsilon$ are exact smooth solutions of $(\mathcal{E})$ and they let ε goes to zero. Yet, this construction is of a very special form. First, it comes from shear layers (these are steady 2D solutions) as

$$\widetilde{\boldsymbol{u}}_s^\varepsilon(t,x) = \widetilde{\boldsymbol{u}}_s^\varepsilon(0,x) = {}^t\big(\boldsymbol{g}(x_2,\varepsilon^{-1}x_2), 0\big) \in \mathbb{R}^2.$$

Secondly, it involves a phase $\varphi_0(t,x) \equiv x_2$ which does not depend on ε. Of course, this is a common fact [11], [21], [20], [28] when dealing with such large amplitude high frequency waves. Nevertheless, this is far from giving a complete idea of what can happen. Our aim in this paper is to develop a theory which allows to remove the two restrictions mentioned above.

Section 2 is devoted to notations.

Section 3 gives the main results.

Subsection 3.1 states Theorem 3.1. Introduce the *geometrical* phase

$$\varphi_g^\varepsilon(t,x) := \varphi_0(t,x) + \sum_{k=1}^{\ell-1} \varepsilon^{k/\ell}\varphi_k(t,x), \quad \ell \in \mathbb{N}_*.$$

Fix $\flat = (\ell, N) \in \mathbb{N}^2$ where the integers ℓ and N are such that $0 < \ell < N$. Theorem 3.1 provides with *approximate* solutions $\boldsymbol{u}_\flat^\varepsilon$ defined on the interval $[0,T]$ with $T > 0$ and having the form

$$\begin{aligned}\boldsymbol{u}_\flat^\varepsilon(t,x) &= {}^t(\boldsymbol{u}_\flat^{\varepsilon 1}, \dots, \boldsymbol{u}_\flat^{\varepsilon d})(t,x) \\ &= \boldsymbol{u}_0(t,x) + \sum_{k=1}^{N} \varepsilon^{k/\ell} U_k\big(t,x,\varepsilon^{-1}\varphi_g^\varepsilon(t,x)\big)\end{aligned} \tag{1.2}$$

where the smooth profiles

$$U_k(t,x,\theta) = {}^t(U_k^1,\dots,U_k^d)(t,x,\theta) \in \mathbb{R}^d, \quad 1 \le k \le N$$

are periodic functions of $\theta \in \mathbb{T} := \mathbb{R}/\mathbb{Z}$. We assume that

$$\exists (t,x,\theta) \in [0,T] \times \mathbb{R}^d \times \mathbb{T}, \quad \partial_\theta U_1(t,x,\theta) \neq 0.$$

The family $\{\boldsymbol{u}_\flat^\varepsilon\}_{\varepsilon\in]0,1]}$ is ε-stratified [20] with respect to the phase φ_g^ε with in general $\varphi_g^\varepsilon \not\equiv \varphi_0$. The presence in φ_g^ε of the non trivial functions φ_k with $1 \le k \le \ell - 1$ is necessary and sufficient to encompass all the *geometrical* features of the propagation.

We say that $\{\boldsymbol{u}_\flat^\varepsilon\}_\varepsilon$ is a *weak*, a *strong* or a *turbulent* oscillation according as we have respectively $\ell = 1$, $\ell = 2$ or $\ell \ge 3$. The order of magnitude of the energy of the oscillations is $\varepsilon^{1/\ell}$. Compute the vorticities associated with $\boldsymbol{u}_\flat^\varepsilon$. These are the skew-symmetric matrices $\Omega_\flat^\varepsilon = (\Omega_{\flat j}^{\varepsilon i})_{1\le i,j\le d}$ where

$$\begin{aligned}
\Omega_{\flat j}^{\varepsilon i}(t,x) &:= (\partial_j \boldsymbol{u}_\flat^{\varepsilon i} - \partial_i \boldsymbol{u}_\flat^{\varepsilon j})(t,x) \\
&= \sum_{k=1}^N \varepsilon^{k/\ell - 1} (\partial_j \varphi_g^\varepsilon \partial_\theta U_k^i - \partial_i \varphi_g^\varepsilon \partial_\theta U_k^j)\big(t,x,\varepsilon^{-1}\varphi_g^\varepsilon(t,x)\big) \\
&\quad + (\partial_j \boldsymbol{u}_0^i - \partial_i \boldsymbol{u}_0^j)(t,x) + \sum_{k=1}^N \varepsilon^{k/\ell} (\partial_j U_k^i - \partial_i U_k^j)\big(t,x,\varepsilon^{-1}\varphi_g^\varepsilon(t,x)\big).
\end{aligned}$$

The principal term in $\Omega_\flat^\varepsilon$ is of size $\varepsilon^{1/\ell-1}$. When $\ell \ge 2$, there is no uniform majoration in L^p on the family $\{\Omega_\flat^\varepsilon\}_{\varepsilon\in]0,1]}$ since

$$\lim_{\varepsilon\to 0} \|\Omega_\flat^\varepsilon\|_{L^p([0,T]\times\mathbb{R}^d)} = +\infty, \quad \forall p \in [1,\infty].$$

In particular, if $d = 3$, there is no uniform control on the enstrophy

$$\int_0^T \int_{\mathbb{R}^3} |\omega_\flat^\varepsilon(t,x)|^2 \,\mathrm{d}t\,\mathrm{d}x, \quad \omega_\flat^\varepsilon(t,x) := (\nabla \wedge \boldsymbol{u}_\flat^\varepsilon)(t,x) \equiv \Omega_\flat^\varepsilon(t,x).$$

We see here that strong and turbulent oscillations are examples of quasi-singularities. Observe that the expansion (1.2) involves a more complicated structure than in (1.1) though the corresponding regime is less singular.

Subsection 3.2 states the Proposition 3.1. Introduce the *complete* phase

$$\varphi_\flat^\varepsilon(t,x) := \varphi_0(t,x) + \sum_{k=1}^N \varepsilon^{k/\ell} \varphi_k(t,x).$$

Proposition 3.1 deals with *approximate* solutions $\widetilde{\boldsymbol{u}}_\flat^\varepsilon$ defined on the interval $[0,T]$ with $T > 0$ and having the form

$$\begin{aligned}
\widetilde{\boldsymbol{u}}_\flat^\varepsilon(t,x) &= {}^t(\widetilde{\boldsymbol{u}}_\flat^{\varepsilon 1},\dots,\widetilde{\boldsymbol{u}}_\flat^{\varepsilon d})(t,x) \\
&= \boldsymbol{u}_0(t,x) + \sum_{k=1}^N \varepsilon^{k/\ell} \widetilde{U}_k\big(t,x,\varepsilon^{-1}\varphi_\flat^\varepsilon(t,x)\big)
\end{aligned} \tag{1.3}$$

where the smooth profiles

$$\widetilde{U}_k(t,x,\theta) = {}^t(\widetilde{U}_k^1,\dots,\widetilde{U}_k^d)(t,x,\theta) \in \mathbb{R}^d, \quad 1 \le k \le N$$

are periodic functions of $\theta \in \mathbb{T}$. Again

$$\exists (t,x,\theta) \in [0,T] \times \mathbb{R}^d \times \mathbb{T}, \quad \partial_\theta \widetilde{U}_1(t,x,\theta) \neq 0.$$

Section 4 shows at first Proposition 3.1 and then Theorem 3.1.

The proof of Proposition 3.1 is based on some induction argument which is quite straightforward. In fact, the difficulty is hidden in the introduction of the *adjusting* phase

$$\varphi_a^\varepsilon(t,x) := \varepsilon^{-1}(\varphi_\flat^\varepsilon - \varphi_g^\varepsilon)(t,x) = \sum_{k=l}^{N} \varepsilon^{k/\ell-1}\varphi_k(t,x).$$

Indeed, the use of the geometrical phase φ_g^ε does not suffice to perform the BKW analysis. Among other things, the extra terms φ_k with $\ell \le k \le N$ must be incorporated in order to put the system of formal equations in a triangular form.

Subsection 4.2 explains how to deduce Theorem 3.1 from Proposition 3.1. It mainly consists in eliminating the adjusting phase (and in checking that the remainder created by that operation is small) as well as in replacing the small divergence of Proposition 3.1 by a zero divergence.

Section 5 interprets the results 3.1.

It starts with various comments related to the Leray projector, the infinite accuracy of approximate solutions, the finite speed of propagation and the large time existence.

Subsection 5.2 proceeds to a careful study of the hierarchy of phases. We examine successively the phase shift φ_1, the phase shift φ_2, and the other terms φ_k with $3 \le k \le N$.

The formal construction reveals that the phase shift φ_1 and the terms φ_k with $2 \le k \le \ell - 1$ play different parts. The rôle of φ_1 is partly revealed in the articles [7] and [8] which deal with the case $\ell = 2$. When $\ell \ge 3$, the phenomenon to emphasize is the creation of the φ_k with $2 \le k \le \ell - 1$. Indeed, suppose that

$$\varphi_2(0,.) \equiv \cdots \equiv \varphi_{\ell-1}(0,.) \equiv 0, \quad \ell \ge 3.$$

Then, generically, we find

$$\exists t \in \,]0,T], \quad \varphi_2(t,.) \not\equiv 0, \ \dots, \ \varphi_{\ell-1}(t,.) \not\equiv 0.$$

Now, starting with *large* amplitude waves (this corresponds to the limit case $\ell = +\infty$) that is

$$\boldsymbol{u}_\infty^\varepsilon(0,x) = \sum_{k=0}^{\infty} \varepsilon^k U_k\big(0,x,\varepsilon^{-1}\varphi_0(0,x)\big), \quad \partial_\theta U_0 \not\equiv 0,$$

the description of $\boldsymbol{u}^{\varepsilon}_{\infty}(t,.)$ on the interval $[0,T]$ with $T>0$ needs the introduction of an *infinite cascade* of phases φ_k. The scenario is the following. Oscillations of the velocity develop spontaneously in all the intermediate frequencies $\varepsilon^{k/\ell-1}$ and in all the corresponding directions $\nabla\varphi_k(t,x)$. This expresses *turbulent* features in the flow.

Subsection 5.3 alludes to *closure* problems. This is the classical difficulty encountered when dealing with expansions as $\boldsymbol{u}^{\varepsilon}_{\flat}$. It is solved here through the introduction of the φ_k with $1\leq k\leq N$.

Subsection 5.4 insists on *obvious* instabilities which are mechanisms of amplification which can be detected just by looking at the BKW analysis presented before. It allows to retrieve known non linear instability results on Euler equations (see Proposition 5.1).

Subsection 5.5 and subsection 5.6 are mainly heuristical. They could also interest researchers in Fluid mechanics. They contain no precise statement or proof but consist in reading Theorem 3.1 in the light of previous numerical, mathematical or physical results. They derive many informations about microstructures, compensated compactness and non linear geometric optics. They also confirm observations which have been made in the statistical approach of turbulences [16], [24].

Section 6 consider parabolic perturbations of Euler equations. This change of framework has two main motivations.

First, it has a physical meaning. Most real models involve some viscosity. And, even if it were only at a formal level, it is interesting to determine what is the size and the structure of the dissipation terms which could be incorporated without changing the phenomena under study.

Secondly, it has implications on the *stability*. The expressions $\boldsymbol{u}^{\varepsilon}_{\flat}$ are only approximate solutions of Euler equations, yielding small error terms $\boldsymbol{f}^{\varepsilon}_{\flat}$ as source terms. The matter is to know if the addition of (well-adjusted) dissipation terms implies the existence of exact solutions (of Navier-Stokes type equations) which coincide with $\boldsymbol{u}^{\varepsilon}_{\flat}(0,.)$ at time $t=0$, which are defined on $[0,T]$ where $T>0$ is independent on ε, and which are close to approximate divergence free solutions like $\boldsymbol{u}^{\varepsilon}_{\flat}$.

These two directions are difficult tasks. In this paper, we will be satisfied to touch on these subjects.

In Subsection 6.1, we build (Proposition 6.1) approximate solutions $\{u^{\varepsilon}_{\flat}\}_{\varepsilon}$ to some Navier-Stokes equation $(\mathcal{NS})$ involving the variables t, x and θ. We start by describing the properties of the parabolic perturbation. The viscosity is vanishing and anisotropic. It could be real or artificial but it must be *compatible* with the complete phase $\varphi^{\varepsilon}_{\flat}$. Then, we adapt the proof of subsection 4.1 to this new setting. In particular, we are faced with the study of the divergence free relation written in the (t,x,θ) variables.