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A CLASS OF NON-RATIONAL SURFACE SINGULARITIES WITH BIJECTIVE NASH MAP

by Camille Plénat & Patrick Popescu-Pampu

Abstract. — Let $(\mathcal{S}, 0)$ be a germ of complex analytic normal surface. On its minimal resolution, we consider the reduced exceptional divisor E and its irreducible components E_i, $i \in I$. The Nash map associates to each irreducible component C_k of the space of arcs through 0 on $\mathcal{S}$ the unique component of E cut by the strict transform of the generic arc in C_k. Nash proved its injectivity and asked if it was bijective. As a particular case of our main theorem, we prove that this is the case if $E \cdot E_i < 0$ for any $i \in I$.

Résumé (*Une classe de singularités non-rationnelles de surfaces ayant une application de Nash bijective*)

Soit $(\mathcal{S}, 0)$ un germe de surface analytique complexe normale. Nous considérons le diviseur exceptionnel réduit E et ses composantes irréductibles E_i, $i \in I$ sur sa résolution minimale. L'application de Nash associe à chaque composante irréductible C_k de l'espace des arcs passant par 0 sur $\mathcal{S}$, l'unique composante de E rencontrée par la transformée stricte de l'arc générique dans C_k. Nash a prouvé son injectivité et a demandé si elle était bijective. Nous prouvons que c'est le cas si $E \cdot E_i < 0$ pour tout $i \in I$ comme cas particulier de notre théorème principal.

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1. Introduction

Let $(\mathcal{S},0)$ be a germ of complex analytic normal surface. Let

$$\pi_m : (\widetilde{\mathcal{S}}_m, E) \longrightarrow (\mathcal{S},0)$$

be its minimal resolution, where E is the reduced exceptional divisor of π_m, and let $(E_i)_{i\in I}$ be the irreducible components of E. A resolution is called *good* if E has normal crossings and if all its components are smooth. It is important to notice that, by Grauert's Contractibility Theorem for surfaces [5], there exist singularities whose minimal resolution is not good.

An *arc through* 0 on $\mathcal{S}$ is a germ of formal map $(\mathbb{C},0) \to (\mathcal{S},0)$.

We denote by $(\mathcal{S},0)_\infty$ the *space of arcs* through 0 on $\mathcal{S}$. It can be canonically given the structure of a scheme over $\mathbb{C}$, as the projective limit of schemes of finite type obtained by truncating arcs at each finite order. So it makes sense to speak about its irreducible components $(C_k)_{k\in K}$. For each arc represented by an element in C_k, one can consider the intersection point with E of its strict transform on $\widetilde{\mathcal{S}}_m$. For a generic element of C_k (in the Zariski topology), this point is situated on a unique irreducible component of E. In this manner one defines a map

$$\mathcal{N} : \big\{C_k \mid k \in K\big\} \longrightarrow \big\{E_i \mid i \in I\big\}$$

which is called *the Nash map* associated to the germ $(\mathcal{S},0)$. It was defined by Nash around 1966, in a preprint published later as [17]. He proved that the map $\mathcal{N}$ is injective (which shows in particular that K is a finite set) and asked the question:

Is the map $\mathcal{N}$ bijective?

This question is now called *the Nash problem on arcs.* No germ $(\mathcal{S},0)$ is known for which the answer is negative. But the bijectivity of $\mathcal{N}$ was only proved till now for special classes of singularities:

- for the germs of type $(\mathbb{A}_n)_{n\geq 1}$ by Nash himself in [17];
- for normal minimal singularities by Reguera [21]; different proofs were given by Plénat [19] and by Fernández-Sánchez [4];
- for sandwiched singularities it was sketched by Reguera [22], using her common work [15] with Lejeune-Jalabert on the wedge problem;
- for the germs of type $(\mathbb{D}_n)_{n\geq 4}$ by Plénat [19];
- for the germs with a good $\mathbb{C}^*$-action such that the curve $\operatorname{Proj}\mathcal{S}$ is not rational, it follows immediately by combining results of Lejeune-Jalabert [13] and Reguera [22].

With the exception of the last class, all the other ones consist only in *rational* singularities and can be defined purely topologically.

Here we prove that the Nash map is bijective for a new class of surface singularities (Theorem 5.1), whose definition depends only on the intersection matrix of the minimal resolution and not on the genera or possible singularities of the components E_i. In particular, their minimal resolution may not be good, which contrasts with the classes of singularities described before. The following (Corollary 5.2) is a particular case of the main theorem:

If $E \cdot E_i < 0$ *for any* $i \in I$*, then the Nash map* $\mathcal{N}$ *is bijective.*

We also show (Corollary 5.5) that the hypothesis of the previous corollary are verified by an infinity of topologically pairwise distinct *non-rational* singularities, which explains the title of the article.

The Nash map can also be defined in higher dimensions, over any field, for not necessarily normal schemes which admit resolutions of their singularities. It is always injective and the same question can also be asked. Ishii and Kollár proved in [6] (a good source for everything we use about spaces of arcs and the Nash map, as well as for references on related works) that it is not always bijective. Indeed, they gave a counterexample in dimension 4, which can be immediately transformed in a counterexample in any larger dimension. They left open the cases of dimensions 2 and 3...

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2. A criterion for distinguishing components of the space of arcs

Consider a germ $(\mathcal{S}, 0)$ of normal surface and its minimal resolution morphism

$$\pi_m : (\widetilde{\mathcal{S}}_m, E) \longrightarrow (\mathcal{S}, 0).$$

If D is a divisor on $\widetilde{\mathcal{S}}_m$, it can be uniquely written as the sum of a divisor supported by E – called *the exceptional part* of D – and a divisor whose support meets E in a finite number of points. If D consists only of its exceptional part, we say that D is *purely exceptional*.

For each $i \in I$, let v_{E_i} be the divisorial valuation defined by E_i on the fraction field of the analytic local ring $\mathcal{O}_{\mathcal{S},0}$. Denote by $\mathfrak{m}_{\mathcal{S},0}$ the maximal ideal of this local ring. If $f \in \mathfrak{m}_{\mathcal{S},0}$, the exceptional part of $\operatorname{div}(f \circ \pi_m)$ is precisely

$$\sum_{i \in I} v_{E_i}(f) E_i.$$

For each component E_i of E, consider the arcs on $\widetilde{\mathcal{S}}_m$ whose closed points are on $E_i - \bigcup_{j \neq i} E_j$ and which intersect E_i transversally. Consider the set of their images in $(\mathcal{S},0)_\infty$ and denote its closure by $V(E_i)$. The sets $V(E_i)$ are irreducible and (see Lejeune-Jalabert [14, Appendix 3])

$$(S,0)_\infty = \bigcup_{i\in I} V(E_i).$$

The following proposition is a special case of a general one proved by the first author in [18] (see also [19]) for non-necessarily normal germs of any dimension and for arbitrary resolutions. It generalizes an equivalent result proved by Reguera [21, Thm. 1.10] for the case of rational surface singularities.

Proposition 2.1. — *If there exists a function $f \in \mathfrak{m}_{\mathcal{S},0}$ such that $v_{E_i}(f) < v_{E_j}(f)$, then $V(E_i) \nsubseteq V(E_j)$.*

Proof. — Let $(\mathcal{S},0) \hookrightarrow (\mathbb{C}^n,0)$ be an analytic embedding of the germ $(\mathcal{S},0)$. Denote by $(x_1,\dots,x_n)$ the coordinates of $\mathbb{C}^n$. An arc $\phi \in (\mathcal{S},0)_\infty$ is then represented by n formal power series

$$\Big(x_k(t) = \sum_{\ell=1}^{\infty} a_{k,\ell}\, t^\ell\Big)_{1\le k\le n},$$

where the coefficients $(a_{k,\ell})_{k,\ell}$ are subjected to algebraic constraints, coming from the fact that the arc must lie on $\mathcal{S}$.

For each $j \in I$, a Zariski open set $U_f(E_j)$ in $V(E_j)$ consists of the images by π_m of the arcs on $\widetilde{\mathcal{S}}_m$ which meet transversely E_j in a smooth point of $\operatorname{div}(f\circ\pi_m)$. If $\phi \in U_f(E_j)$, we have

$$v_{E_j}(f) = v_t(f\circ\phi)$$

where $v_t(g)$ denotes the order in t of $g \in \mathbb{C}[[t]]$.

This shows that the first $v_{E_j}(f) - 1$ coefficients of $f\circ\phi$, seen as elements of $\mathbb{C}[a_{k,\ell}]_{k,\ell}$, must vanish. Their vanishing defines a closed subscheme $Z_{f,j}$ of $(\mathcal{S},0)_\infty$. Therefore, $U_f(E_j) \subset Z_{f,j}$, which implies that

$$V(E_j) \subset Z_{f,j}.$$

As $v_{E_i}(f) < v_{E_j}(f)$, we see that no element of $U_f(E_i)$ is included in $Z_{f,j}$, which shows that

$$V(E_i) \nsubseteq Z_{f,j}.$$

The proposition follows. □

3. Construction of functions with prescribed divisor

In this section, $\pi : (\widetilde{\mathcal{S}},E) \to (\mathcal{S},0)$ denotes *any* resolution of $(\mathcal{S},0)$.

Inside the free abelian group generated by $(E_i)_{i\in I}$ we consider the set

$$\mathcal{L}(\pi) := \big\{D \mid D \neq 0,\ D\cdot E_i \le 0, \forall i \in I\big\}.$$

It is a semigroup with respect to addition, which we call (following Lê [11, 3.2.5]) *the Lipman semigroup* associated to π (see Lipman [16, §18]). It is known that it consists only of effective divisors (see Lipman [16, §18 (ii)]).

We call *strict Lipman semigroup* of π the subset:

$$\mathcal{L}^0(\pi) := \big\{D \in \mathcal{L}(\pi) \mid D \cdot E_i < 0,\ \forall i \in I\big\}$$

of the Lipman semigroup of π. It is always non-empty.

The importance of the Lipman semigroup comes from the fact that the exceptional parts of the divisors of the form $\operatorname{div}(f \circ \pi)$, where $f \in \mathfrak{m}_{\mathcal{S},0}$, are elements of it. The converse is true for rational surface singularities, but this is not the case for arbitrary surface singularities.

We give now a numerical criterion on a divisor $D \in \mathcal{L}(\pi)$ which allows one to conclude that it is the exceptional part of a divisor of the form $\operatorname{div}(f \circ \pi)$:

Proposition 3.1. — *Let D be an effective purely exceptional divisor, such that for any $i, j \in I$, one has the inequality:*

$$(D + E_i + K_{\widetilde{\mathcal{S}}}) \cdot E_j + 2\delta_i^j \leq 0$$

where δ_i^j is Kronecker's symbol. Then there exists a function $f \in \mathfrak{m}_{\mathcal{S},0}$ such that the exceptional part of $\operatorname{div}(f \circ \pi)$ *is precisely D.*

Proof. — We use the following Grauert-Riemenschneider type vanishing theorem, proved by Laufer [9] for analytic germs and by Ramanujam [20] for algebraic ones (see also Bădescu [1, 4.1]):

If L is a divisor on $\widetilde{\mathcal{S}}$ such that $L \cdot E_j \geq K_{\widetilde{\mathcal{S}}} \cdot E_j$ for all $j \in I$, then

$$H^1\big(\mathcal{O}_{\widetilde{\mathcal{S}}}(L)\big) = 0.$$

We apply the theorem to $L = -D - E_i$, for any $i \in I$. Our hypothesis implies that $H^1(\mathcal{O}_{\widetilde{\mathcal{S}}}(-D-E_i)) = 0$. Then, from the exact cohomology sequence associated to the short exact sequence of sheaves

$$0 \to \mathcal{O}_{\widetilde{\mathcal{S}}}(-D - E_i) \longrightarrow \mathcal{O}_{\widetilde{\mathcal{S}}}(-D) \xrightarrow{\psi_i} \mathcal{O}_{E_i}(-D) \to 0$$

we deduce the *surjectivity* of the restriction map

$$\psi_{i*} : H^0\big(\mathcal{O}_{\widetilde{\mathcal{S}}}(-D)\big) \longrightarrow H^0\big(\mathcal{O}_{E_i}(-D)\big).$$

By Serre duality on the irreducible (possibly singular) curve E_i (see Reid [23, 4.10]), we get

$$h^1\big(\mathcal{O}_{E_i}(-D)\big) = h^0\big(\mathcal{O}_{E_i}(K_{\widetilde{\mathcal{S}}} + E_i + D)\big) = 0.$$

For the last equality we have used the hypothesis $(D + E_i + K_{\widetilde{\mathcal{S}}}) \cdot E_i \leq -2 < 0$, which shows that the line bundle $\mathcal{O}_{E_i}(K_{\widetilde{\mathcal{S}}} + E_i + D)$ cannot have a non-trivial section.