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STEINNESS OF BUNDLES WITH FIBER A REINHARDT BOUNDED DOMAIN

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ABSTRACT. — Let E denote a holomorphic bundle with fiber D and with basis B. Both D and B are assumed to be Stein. For D a Reinhardt bounded domain of dimension $d = 2$ or 3, we give a necessary and sufficient condition on D for the existence of a non-Stein such E (Theorem 1); for $d = 2$, we give necessary and sufficient criteria for E to be Stein (Theorem 2). For D a Reinhardt bounded domain of any dimension not intersecting any coordinate hyperplane, we give a sufficient criterion for E to be Stein (Theorem 3).

RÉSUMÉ (*Fibrés de Stein à fibre un domaine de Reinhardt borné*)
Soit E un fibré holomorphe à fibre D et base B. On suppose que D et B sont de Stein. Si D est un domaine de Reinhardt borné de dimension 2 ou 3, on donne une condition nécessaire et suffisante sur D pour l'existence d'un tel fibré E qui ne soit pas Stein (Théorème 1) ; pour $d = 2$ on donne des conditions nécessaires et suffisantes pour que E soit de Stein (Théorème 2). Si D est un domaine de Reinhardt de dimension quelconque qui n'intersecte pas les hyperplans de coordonnées, on donne un critère suffisant pour que E soit de Stein.

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1. Introduction and notations

Stein manifolds can be characterized by the fact that they holomorphically embed in $\mathbb{C}^N$ for some N. From that point of view it is obvious that if F and B are Stein manifolds, then the product $E = F \times B$ is Stein. More generally, take a fiber bundle E with fiber F and with basis B, which we shall denote by

$$E \xrightarrow{F} B.$$

Is such an E necessarily Stein? That question was asked fifty years ago by J.-P. Serre [10], and is often referred to as "the Serre Problem" in the literature.

A counterexample with $F = \mathbb{C}^2$ was produced by H. Skoda [16] in 1977. On the other hand, many interesting "positive results" have been obtained (see below).

Following [9], we shall say that a Stein manifold F is of class $\mathcal{S}$, or $F \in \mathcal{S}$ for short, when[(1)]

For any bundle $E \xrightarrow{F} B$ with B Stein, E is Stein.

K. Stein [18] proved that if $\dim F = 0$, then $F \in \mathcal{S}$. Building on previous work by A. Hirschowitz [5], Y.T. Siu [13] and N. Sibony [12], N. Mok [7] proved that if $\dim F = 1$, then $F \in \mathcal{S}$. Skoda's above result can be stated as: $\mathbb{C}^2 \notin \mathcal{S}$.

In this paper we focus on the case where F is a bounded domain $D \subset \mathbb{C}^n$ ("domain" means connected open subset). Several results showing that large classes of bounded domains belong to $\mathcal{S}$ have been proved (cf. [14], [17] and [2]). Nevertheless G. Cœuré and J.-J. Lœb [1] produced, for each given $A \in \mathrm{SL}_2(\mathbb{Z})$ with $|\operatorname{trace} A| > 2$, a non Stein bundle

$$E_{\mathrm{CL}} \xrightarrow{D_{\mathrm{CL}}} \mathbb{C}^*$$

whose fiber D_{CL} is a bounded Stein domain subset of $(\mathbb{C}^*)^2$. Thus $D_{\mathrm{CL}} \notin \mathcal{S}$. Their D_{CL} has the following properties (see Figure 1):

▷ D_{CL} has the Reinhardt symmetry, i.e., it is invariant by the map $(z_1, z_2) \mapsto (\alpha_1 z_1, \alpha_2 z_2)$ for any complex numbers α_1 and α_2 of modulus 1;

▷ D_{CL} has an automorphism g of the form $g(z_1, z_2) = (z_1^a z_2^b, z_1^c z_2^d)$ with

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = A.$$

The second named author generalized their construction and gave a better understanding of those bundles. In [20], a key point is the existence of a *g-equivariant* open dense embedding $D_{\mathrm{CL}} \hookrightarrow \widehat{D}_{\mathrm{CL}}$, where $\widehat{D}_{\mathrm{CL}} \setminus D_{\mathrm{CL}}$ is an infinite chain of rational curves. Roughly speaking, the non-Steinness of E_{CL} is "explained" by what happens near that infinite chain.

(1) Conveniently enough, that letter honors simultaneously Serre, Sibony, Siu, Skoda, Stehlé and Stein.

Our goal here is to answer the following "converse" question:

Let $D \subset \mathbb{C}^n$, with $n = 2$ or 3, be any *bounded Stein Reinhardt domain. Does D belong to $\mathcal{S}$?*

In other words, does there exist a bundle $E \xrightarrow{D} B$ with B Stein and E non-Stein? The answer is contained in Theorem 1 below. It reveals that in dimension two, Cœuré-Lœb's examples D_{CL} (for all $A \in \mathrm{SL}_2(\mathbb{Z})$) are essentially the only Reinhardt bounded domains not in $\mathcal{S}$: all other examples are provided by g-invariant subdomains of some D_{CL}. Moreover, it is easily checked that the interior of the closure in $\widehat{D}_{\mathrm{CL}}$ of such a subdomain still contains the infinite chain of curves, so from the point of view of [20], it is natural that those domains do not belong to $\mathcal{S}$. Indeed, proofs in [1] and [20] apply almost verbatim to show that they do not belong to $\mathcal{S}$. We shall see that both methods and results become more complicated in dimension three.

We also address the following question:

Given D bounded and Reinhardt not in $\mathcal{S}$ and B Stein, can we give a characterization of the Steinness of a bundle $E \xrightarrow{D} B$?

For a two-dimensional D, we give in Theorem 2 both a necessary criterion and a sufficient criterion. For a higher dimensional D, we give a partial answer in Theorem 3.

We work throughout the article in the complex category. In other words, all manifolds and maps we deal with are holomorphic. By the word "bundle" we mean a locally trivial holomorphic fiber bundle. We shall also use the notations $\mathcal{O}(E)$ for the set of holomorphic functions on E, and S^1 and Δ will respectively denote the unit circle and the unit disk in $\mathbb{C}$.

N.B.: Most proofs are postponed until the end of the paper, in Section 4.

We shall make use of several known results about a given bundle $E \xrightarrow{D} B$ with B Stein and $D \subset \mathbb{C}^n$ bounded and Stein. Namely:

$\triangleright$ E is a flat bundle (see [6] or [14]).

That means E can be defined by locally constant transition functions.

All flat bundles can be constructed as follows. Let ρ be a morphism $\pi_1(B) \to \mathrm{Aut}(D)$. Such a ρ induces a $\pi_1(B)$-action on D. Denote by $\widetilde{B}$ the universal cover of B, and consider the diagonal action of $\pi_1(B)$ on $\widetilde{B} \times D$. Define

$$E = \frac{\widetilde{B} \times D}{\pi_1(B)} \cdot$$

Then the projection map $\widetilde{B} \times D \to \widetilde{B}$ induces a map $E \to B$ that turns E into a bundle with fiber D. The *structural group* $G_{\mathrm{struct}}(E)$ of E is by definition the image of ρ. That definition is quite improper because $G_{\mathrm{struct}}(E)$ does not only depend on the isomorphism class of E as an F-bundle over B, but also on the ρ chosen. That "subtlety" won't matter for our purposes...

▹ E is holomorphically separable (see [14]).

▹ E is Stein if $G_{\mathrm{struct}}(E)$ is compact or contained in a compact group (see [12] and [14]).

▹ E is Stein if $G_{\mathrm{struct}}(E)$ has finitely many connected components (see [14]).

Given $g \in \mathrm{Aut}(D)$, there is exactly one bundle $E \xrightarrow{D} \mathbb{C}^*$ with monodromy g. Namely, $E = \mathbb{C} \times D/\mathbb{Z}$, where the $\mathbb{Z}$-action is the "diagonal action" generated by

$$\tilde{g} : \mathbb{C} \times D \longrightarrow \mathbb{C} \times D, \quad \big(z\,;d\big) \longmapsto \big(z+1\,; g(d)\big).$$

We shall call that bundle the *complex suspension of g.* It has infinite cyclic $\mathcal{A}(E)$, generated by g.

For simplicity, we introduce the following notations and results assuming $n = 2$, but they all extend in the "obvious" way to any $n \geq 2$.

By a well-known criterion for the Steinness of a Reinhardt domain (see [8]), the map

$$\text{'log'} : (\mathbb{C}^*)^2 \longrightarrow \mathbb{R}^2, \quad (z_1, z_2) \longmapsto \big(\log|z_1|, \log|z_2|\big)$$

induces a one-to-one correspondence between Stein Reinhardt domains of $(\mathbb{C}^*)^2$ and open convex subsets of $\mathbb{R}^2$.

Now take $D \subset (\mathbb{C}^*)^2$ a *bounded* Stein Reinhardt domain. We shall denote by $\log D$ the image of D by the above map. To make more explicit the one-to-one correspondence

$$\log D \longleftrightarrow \; D,$$

remark that D can be recovered from $\log D$ as the image of the "tube" $\log D + i\mathbb{R}^2 \subset \mathbb{C}^2$ by the map

$$\text{'exp'} : (w_1, w_2) \longmapsto (z_1, z_2) = (\mathrm{e}^{w_1}, \mathrm{e}^{w_2}).$$

Moreover $\log D$ contains no affine line: otherwise $\log D + i\mathbb{R}^2$ would contain a copy of $\mathbb{C}$ on which 'exp' would restrict to a non-constant map from $\mathbb{C}$ to D, contradicting the boundedness of D. By [21], the converse statement holds: for a Stein Reinhardt D, if $\log D$ contains no affine line, then D is isomorphic to a bounded domain (we won't use that fact, though).

We denote by $\mathrm{Aut}(D)$ the group of automorphisms of D. By [11],

$$\mathrm{Aut}(D) = \mathrm{Aut}^{\mathbb{R}}_{\mathrm{alg}}(D) \ltimes \mathrm{Aut}_0(D),$$

where

▹ $\mathrm{Aut}_0(D)$ is the connected component of the identity,

▷ $\mathrm{Aut}_{\mathrm{alg}}(D)$ is the subgroup of $\mathrm{Aut}(D)$ defined by: For each $g \in \mathrm{Aut}_{\mathrm{alg}}(D)$, there exist $A_g = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{GL}_2(\mathbb{Z})$ and $\alpha_1, \alpha_2 \in \mathbb{C}^*$ such that

$$g(z_1, z_2) = (\alpha_1 z_1^a z_2^b, \alpha_2 z_1^c z_2^d).$$

(Given g, such A_g, α_1 and α_2 are unique.)

▷ $\mathrm{Aut}^{\mathbb{R}}_{\mathrm{alg}}(D)$ denotes the subgroup of $\mathrm{Aut}_{\mathrm{alg}}(D)$ of all g's with real and positive α_i's. Thus

$$\mathrm{Aut}_{\mathrm{alg}}(D) = \mathrm{Aut}^{\mathbb{R}}_{\mathrm{alg}}(D) \ltimes (S^1)^2,$$

and by Lemma 1.4, $\mathrm{Aut}^{\mathbb{R}}_{\mathrm{alg}}(D)$ is a discrete group.

For $g \in \mathrm{Aut}_{\mathrm{alg}}(D)$, we shall denote by f_g the map

$$f_g : \log D \longrightarrow \log D, \quad p \longmapsto A_g p + b_g,$$

where $b_g = (\log|\alpha_1|, \log|\alpha_2|)$. The correspondence $g \leftrightarrow f_g$ is one-to-one between $\mathrm{Aut}^{\mathbb{R}}_{\mathrm{alg}}(D)$ and the group of affine automorphisms of $\log D$. Remark that

$$f_{g^{-1}}(p) = f_g^{-1}(p) = A_g^{-1} p - A_g^{-1} b_g.$$

Define

$$\mathcal{A}(D) = \left\{ A_g : g \in \mathrm{Aut}^{\mathbb{R}}_{\mathrm{alg}}(D) \right\} \subset \mathrm{GL}_2(\mathbb{Z}).$$

It is useful to think of $\mathcal{A}(D)$ as "the set of matrices that act on D".

For a given bundle $E \xrightarrow{D} B$,

$$\mathcal{A}(E) = \left\{ A_g : g \in \mathrm{Aut}^{\mathbb{R}}_{\mathrm{alg}}(D) \cap \big(\mathrm{Aut}_0(D) \cdot G_{\mathrm{struct}}(E) \big) \right\} \subset \mathcal{A}(D).$$

It is useful to think of $\mathcal{A}(E)$ as "the set of matrices that are used to build E".

For any group of matrices $\mathcal{A}$, we denote

$$\mathrm{Spec}_{\mathbb{C}} \mathcal{A} = \bigcup_{A \in \mathcal{A}} \mathrm{Spec}_{\mathbb{C}} A.$$

We can now state the main results of this paper. They consist of the following theorems, and the geometric description (that follows from Theorem 1) given below. We point out the importance of the set $\mathrm{Spec}_{\mathbb{C}} \mathcal{A}(D)$ to study whether a domain D belongs to $\mathcal{S}$ or not. Theorem 1 in the case $n = 2$ is the main result of [9]. Our proof for that case is simpler.

Theorem 1. — *A bounded Stein Reinhardt domain $D \subset \mathbb{C}^n$ with $n = 2$ or 3 belongs to $\mathcal{S}$ if and only if $\mathrm{Spec}_{\mathbb{C}} \mathcal{A}(D) \subset S^1$. In other words:*

▷ *For $n = 2$, $D \notin \mathcal{S}$ if and only if there exists $A \in \mathcal{A}(D)$ with $\mathrm{Spec}_{\mathbb{C}} A = \{\lambda, \lambda^{-1}\}$, $\lambda \in \mathbb{R} \setminus \{+1, -1\}$.*

▷ *For $n = 3$, $D \notin \mathcal{S}$ if and only if (cf. Lemma 1.2) there exists $A \in \mathcal{A}(D)$ such that either*

(a) $\mathrm{Spec}_{\mathbb{C}} A = \{\lambda_1, \lambda_2, \lambda_3\}$ *with λ_i's pairwise distinct and real, or*

(b) $\mathrm{Spec}_{\mathbb{C}} A = \{1, \lambda, \lambda^{-1}\}$ *with $\lambda \in \mathbb{R} \setminus \{+1, -1\}$.*