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POINTED k-SURFACES

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ABSTRACT. — Let S be a Riemann surface. Let $\mathbb{H}^3$ be the 3-dimensional hyperbolic space and let $\partial_\infty\mathbb{H}^3$ be its ideal boundary. In our context, a Plateau problem is a locally holomorphic mapping $\varphi : S \to \partial_\infty\mathbb{H}^3 = \widehat{\mathbb{C}}$. If $i : S \to \mathbb{H}^3$ is a convex immersion, and if N is its exterior normal vector field, we define the Gauss lifting, $\hat{\imath}$, of i by $\hat{\imath} = N$. Let $\overrightarrow{n} : U\mathbb{H}^3 \to \partial_\infty\mathbb{H}^3$ be the Gauss-Minkowski mapping. A solution to the Plateau problem (S, φ) is a convex immersion i of constant Gaussian curvature equal to $k \in (0, 1)$ such that the Gauss lifting $(S, \hat{\imath})$ is complete and $\overrightarrow{n} \circ \hat{\imath} = \varphi$. In this paper, we show that, if S is a compact Riemann surface, if $\mathcal{P}$ is a discrete subset of S and if $\varphi : S \to \widehat{\mathbb{C}}$ is a ramified covering, then, for all $p_0 \in \mathcal{P}$, the solution $(S\setminus\mathcal{P}, i)$ to the Plateau problem $(S \setminus \mathcal{P}, \varphi)$ converges asymptotically as one tends to p_0 to a cylinder wrapping a finite number, k, of times about a geodesic terminating at $\varphi(p_0)$. Moreover, k is equal to the order of ramification of φ at p_0. We also obtain a converse of this result, thus completely describing complete, constant Gaussian curvature, immersed hypersurfaces in $\mathbb{H}^3$ with cylindrical ends.

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Résumé (*k-surfaces à points*). — Soit S une surface de Riemann. Soit $\mathbb{H}^3$ l'espace hyperbolique de dimension 3 et soit $\partial_\infty\mathbb{H}^3$ son bord à l'infini. Dans le cadre de cet article, un problème de Plateau est une application localement holomorphe $\varphi : S \rightarrow \partial_\infty\mathbb{H}^3 = \hat{\mathbb{C}}$. Si $i : S \rightarrow \mathbb{H}^3$ est une immersion convexe, et si N est son champ de vecteurs normal, on définit $\hat{\imath}$, la relevée de Gauss de i, par $\hat{\imath} = N$. Soit $\overrightarrow{n} : U\mathbb{H}^3 \rightarrow \partial_\infty\mathbb{H}^3$ l'application de Gauss-Minkowski. Une solution au problème de Plateau (S,φ) est une immersion convexe i à courbure gaussienne constante égale à $k \in]0,1[$ telle que sa relevée de Gauss $(S,\hat{\imath})$ soit complète en tant que sous-variété immergée et que $\overrightarrow{n}\circ\hat{\imath} = \varphi$. Dans cet article, on montre que, si S est une surface de Riemannn compacte, si $\mathcal{P}$ est un sous-ensemble discret de S et si $\varphi : S \rightarrow \hat{\mathbb{C}}$ est un revêtement ramifié, alors, pour tout $p_0 \in \mathcal{P}$, la solution $(S \setminus \mathcal{P}, i)$ au problème de Plateau $(S \setminus \mathcal{P}, \varphi)$ converge asymptotiquement vers un cylindre qui s'enroule un nombre fini k de fois autour d'une géodésique ayant $\varphi(p_0)$ pour une de ses extrémités lorsqu'on s'approche de p_0. De plus, k est égale à l'ordre de ramification de φ en p_0. On obtient également une réciproque de ce résultat nous permettant de décrire entièrement les surfaces complètes immergées dans $\mathbb{H}^3$ à courbure gaussienne constante et aux bouts cylindriques.

1. Introduction

In this paper, by establishing a result permitting us to describe the behaviour "at infinity" of surfaces of constant Gaussian curvature immersed in 3-dimensional hyperbolic space, we obtain a complete geometric description of solutions to the Plateau problem for compact Riemann surfaces with marked points.

Let $\mathbb{H}^3$ be 3-dimensional hyperbolic space, and let $\partial_\infty\mathbb{H}^3$ be its ideal boundary (see, for example [1]). The ideal boundary of $\mathbb{H}^3$ may be identified canonically with the Riemann sphere $\widehat{\mathbb{C}}$. In this context, following [4] and [9], we define a *Plateau problem* to be a pair (S,φ) where S is a Riemann surface and $\varphi : S \rightarrow \partial_\infty\mathbb{H}^3$ is a locally conformal mapping (i.e., a locally homeomorphic holomorphic mapping). The Plateau problem (S,φ) is said to be of hyperbolic, parabolic or elliptic type depending on whether S is hyperbolic, parabolic or elliptic respectively.

Let $U\mathbb{H}^3$ be the unitary bundle over $\mathbb{H}^3$. For $i : S \rightarrow \mathbb{H}^3$ an immersion, using the canonical orientation of S, we may define the unit normal exterior vector field N over S. This field is a section of $U\mathbb{H}^3$ over i. We define the *Gauss lifting* $\hat{\imath}$ of i by $\hat{\imath} = \mathsf{N}$. We define a *k-surface* to be an immersed surface $\Sigma = (S,i)$ in $\mathbb{H}^3$ of constant Gaussian curvature k whose Gauss lifting $\widehat{\Sigma} = (S,\hat{\imath})$ is a complete immersed surface in $U\mathbb{H}^3$. For $k \in (0,1)$, a solution to the Plateau problem (S,φ) is a k-surface $\Sigma = (S,i)$ such that, if we denote by $\overrightarrow{n}$ the Gauss-Minkowski mapping of $\mathbb{H}^3$, then the Gauss lifting $\hat{\imath}$ of i satisfies

$$\varphi = \overrightarrow{n} \circ \hat{\imath}.$$

In [9] we show that, if (S, φ) is a hyperbolic Plateau problem, then, for all $k \in (0,1)$ there exists a unique solution i to the Plateau problem (S, φ) with constant Gaussian curvature k. Moreover, we show that i depends continuously on φ. In this paper, following on from these ideas, we study the structure of solutions to the Plateau problem (S, φ) when S is a compact Riemann surface with isolated marked points.

The following result, which provides the key to the rest of the paper, describes the behaviour "at infinity" of solutions to the Plateau problem.

Theorem 1.1 (Boundary Behaviour Theorem). — *Let S be a hyperbolic Riemann surface and let $\varphi : S \rightarrow \widehat{\mathbb{C}}$ be a locally conformal mapping. For $k \in (0,1)$, let $i : S \rightarrow U\mathbb{H}^3$ be an immersion such that (S, i) is the unique solution to the Plateau problem (S, φ) with constant Gaussian curvature k. Let K be a compact subset of S and let Ω be a connected component of $S \setminus K$. Let q be an arbitrary point in the boundary of $\varphi(\Omega)$ that is not in $\varphi(\overline{\Omega} \cap K)$.*

If $(p_n)_{n \in \mathbb{N}} \in \Omega$ is a sequence of points such that $(\varphi(p_n))_{n \in \mathbb{N}}$ tends towards q, then the sequence $(i(p_n))_{n \in \mathbb{N}}$ also tends towards q.

Remark. — This theorem confirms our intuition concerning solutions to the Plateau problem. In particular, if S is a Jordan domain in $\partial_\infty \mathbb{H}^3$, if φ is the canonical embedding and if $i : S \rightarrow \mathbb{H}^3$ is a solution to the Plateau problem (S, φ), then the ideal boundary of the immersed surface (S, i) coincides with ∂S.

We use this theorem to study the behaviour of solutions to the Plateau problem near to isolated singularities. We begin by a series of definitions concerning tubes about geodesics. For Γ a geodesic in $\mathbb{H}^3$, we define N_Γ to be the normal bundle over Γ in $U\mathbb{H}^3$:

$$N_\Gamma = \left\{ n_p \in U\mathbb{H}^3 \text{ s.t. } p \in \Gamma,\, n_p \perp T_p\Gamma \right\}.$$

A *tube* about Γ is a pair $T = (S, \hat{\imath})$ where S is a complete surface and $\hat{\imath} : S \rightarrow N_\Gamma$ is a covering map. Since N_Γ is conformally equivalent to $S^1 \times \mathbb{R}$, where S^1 is the circle of radius 1 in $\mathbb{C}$, we may assume either that $S = S^1 \times \mathbb{R}$ or that $S = \mathbb{R} \times \mathbb{R}$. In the former case, $\hat{\imath}$ is a covering map of finite order, and, if k is the order of $\hat{\imath}$, then we say that the tube T is a *tube of order k*. The application $\hat{\imath}$ is then unique up to vertical translations and horizontal rotations of $S^1 \times \mathbb{R}$. In the latter case, we say that T is a *tube of infinite order*. The application $\hat{\imath}$ is then unique up to translations of $\mathbb{R} \times \mathbb{R}$. In the sequel, we will only be interested in tubes of finite order.

Let S be a compact surface and let $\mathcal{P}$ be a finite set of points in S. Let $\hat{\imath} : S \setminus \mathcal{P} \rightarrow U\mathbb{H}^3$ be an immersion. Let p be an arbitrary point in $\mathcal{P}$. We say that $(S \setminus \mathcal{P}, \hat{\imath})$ is *asymtotically tubular* of order k about p if and only if it is a bounded graph over a half tube of order k in $U\mathbb{H}^3$, which tends towards the tube itself as one tends towards infinity. More precisely, let Exp : $TU\mathbb{H}^3 \rightarrow U\mathbb{H}^3$

be the exponential mapping and let NN_Γ be the normal bundle of N_Γ. Then $(S \setminus \mathcal{P}, \hat{\imath})$ is asymptotically tubular of order k about p if there exists

(i) a geodesic Γ and a tube $T = (S^1 \times \mathbb{R}, \hat{\jmath})$ of order k about Γ,

(ii) a section λ of $\hat{\jmath}^* NN_\Gamma$ over $S^1 \times (0, \infty)$,

(iii) a neighbourhood Ω of p in S such that $\mathcal{P} \cap \Omega = \{p\}$, and

(iv) a diffeomorphism $\alpha : S^1 \times (0, \infty) \to \Omega \setminus \{p\}$,

such that

(i) $\hat{\imath} \circ \alpha = \mathrm{Exp} \circ \lambda$,

(ii) $\alpha(\mathrm{e}^{i\theta}, t) \to p$ as $t \to \infty$, and

(iii) for all $p \in \mathbb{N}$, the derivative $D^p \lambda(\mathrm{e}^{i\theta}, t)$ tends to zero as t tends to $+\infty$.

We now obtain the following result.

Theorem 1.2. — *Let S be a Riemann surface. Let $\mathcal{P}$ be a discrete subset of S such that $S \setminus \mathcal{P}$ is hyperbolic. Let $\varphi : S \to \widehat{\mathbb{C}}$ be a ramified covering having critical points in $\mathcal{P}$. Let κ be a real number in $(0, 1)$. Let $i : S \setminus \mathcal{P} \to \mathbb{H}^3$ be the unique solution to the Plateau problem $(S \setminus \mathcal{P}, \varphi)$ with constant Gaussian curvature κ. Let $\widehat{\Sigma} = (S \setminus \mathcal{P}, \hat{\imath})$ be the Gauss lifting of Σ. Let p_0 be an arbitrary point in $\mathcal{P}$.*

If φ has a critical point of order k at p_0, then $\widehat{\Sigma}$ is asymptotically tubular of order k at p_0.

Remark. — This means that if the mapping φ has a critical point of order k at p_0, and is thus equivalent to $z \mapsto z^k$, then the immersed surface $(S \setminus \mathcal{P}, i)$ wraps k times about a geodesic which terminates at $\varphi(p_0)$. We observe that critical points of order 1 are admitted, even though they are not, strictly speaking, critical points.

We also obtain a converse to this result:

Theorem 1.3. — *Let S be a surface and let $\mathcal{P} \subseteq S$ be a discrete subset of S. Let $i : S \setminus \mathcal{P} \to \mathbb{H}^3$ be an immersion such that $\Sigma = (S \setminus \mathcal{P}, i)$ is a k-surface (and is thus the solution to a Plateau problem). Let $\overrightarrow{n} : U\mathbb{H}^3 \to \partial_\infty \mathbb{H}^3$ be the Gauss-Minkowski mapping which sends $U\mathbb{H}^3$ to $\partial_\infty \mathbb{H}^3$. Let $\hat{\imath}$ be the Gauss lifting of i so that $\varphi = \overrightarrow{n} \circ \hat{\imath}$ defines the Plateau problem to which i is the solution. Let $\mathcal{H}$ be the holomorphic structure generated over $S \setminus \mathcal{P}$ by the local homeomorphism φ. Let p_0 be an arbitrary point in $\mathcal{P}$, and suppose that Σ is asymptotically tubular of order k about p_0.*

Then there exists a unique holomorphic structure $\widetilde{\mathcal{H}}$ over $(S \setminus \mathcal{P}) \cup \{p_0\}$ and a unique holomorphic mapping $\tilde{\varphi} : (S \setminus \mathcal{P}) \cup \{p_0\} \to \widehat{\mathbb{C}}$ such that $\widetilde{\mathcal{H}}$ and $\tilde{\varphi}$ extend $\mathcal{H}$ and φ respectively. Moreover, $\tilde{\varphi}$ has a critical point of order k at p_0.

Remark. — Together, these two theorems provide a complete geometric descrition of solutions to the Plateau problem (S, φ) when S is a compact Riemann surface with a finite number of marked points.

Throughout this paper, we will use the convention that $0 \notin \mathbb{N}$.

In the first section, we provide an overview of the definitions and notations that will be used in the sequel. In the second section, we study the differential geometry of the unitary bundle of a Riemannian manifold, focusing, in particular, on the canonical contact and complex structures of this bundle. In the third section, we define the Plateau problem, providing various auxiliary definitions and recalling existing results of [4] and [9] which will be required in the sequel. In the fifth section, we prove Theorem 1.1. In the sixth section, we study the geometry of the Plateau problem $(\mathbb{D}^*, z \mapsto z)$, which provides a model for the study of all other cases. In the seventh section, we prove Theorem 1.2, and in the final section we prove Theorem 1.3.

These results provoke the following reflections concerning potential future avenues of research: first, we obtain a homeomorphism between the space of meromorphic mappings over compact Riemann surfaces with a finite number of marked points on the one hand and complete positive pseudo-holomorphic curves immersed in $U\mathbb{H}^3$ with cylindrical ends on the other. These pseudo-holomorphic curves project down to surfaces of constant Gaussian curvature immersed into $\mathbb{H}^3$. Such an equivalence may well permit us to better understand the structure of either one or both of these two spaces. Second, by integrating primitives of the canonical volume form of $\mathbb{H}^3$ over these immersed surfaces, one obtains a "volume" bounded by these surfaces. If this volume can be shown to be finite, then we would obtain a new function over the Teichmüller space of compact Riemann surfaces with marked points. We would then be interested in the properties of such a function. Finally, since the reasoning employed is essentially geometric in nature, and does not appear to rely on the precise analytic structure of $\mathbb{H}^3$, it seems reasonable to expect an analogous result in the case where $\mathbb{H}^3$ is replaced by a Hadamard manifold whose curvature lies in the range $[-K, -k]$, where $K \geqslant k > 0$ are two positive real numbers.

I would like to thank François Labourie for having initially brought my attention to this problem.

2. Immersed surfaces – Definitions and notations

2.1. Definitions. — In this section we will review basic definitions from the theory of immersed submanifolds and establish the notations that will be used throughout this article.