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CUTTING THE LOSS OF DERIVATIVES FOR SOLVABILITY UNDER CONDITION (Ψ)

by Nicolas Lerner

Abstract. — For a principal type pseudodifferential operator, we prove that condition (ψ) implies local solvability with a loss of $3/2$ derivatives. We use many elements of Dencker's paper on the proof of the Nirenberg-Treves conjecture and we provide some improvements of the key energy estimates which allows us to cut the loss of derivatives from $\epsilon + 3/2$ for any $\epsilon > 0$ (Dencker's most recent result) to $3/2$ (the present paper). It is already known that condition (ψ) does *not* imply local solvability with a loss of 1 derivative, so we have to content ourselves with a loss > 1.

Résumé (*Diminution de la perte de dérivées pour la résolubilité sous la condition* (Ψ))
Pour un opérateur de type principal, nous démontrons que la condition (Ψ) implique la résolubilité locale avec perte de $3/2$ dérivées. Nous utilisons beaucoup d'éléments de la démonstration par Dencker de la conjecture de Nirenberg-Treves et nous limitons la perte de dérivées à $3/2$, améliorant le résultat le plus récent de Dencker (perte de $\epsilon + 3/2$ dérivées pour tout $\epsilon > 0$). La condition (Ψ) n'impliquant pas la résolubilité locale avec perte d'une dérivée, nous devons nous contenter d'une perte > 1.

1. Introduction and statement of the results

1.1. Introduction. — In 1957, Hans Lewy [25] constructed a counterexample showing that very simple and natural differential equations can fail to have

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local solutions; his example is the complex vector field $L_0 = \partial_{x_1} + i\partial_{x_2} + i(x_1 + ix_2)\partial_{x_3}$ and one can show that there exists some C^∞ function f such that the equation $L_0 u = f$ has no distribution solution, even locally. A geometric interpretation and a generalization of this counterexample were given in 1960 by L. Hörmander in [10] and extended in [11] to pseudodifferential operators. In 1970, L. Nirenberg and F. Treves ([29, 30, 31]), after a study of complex vector fields in [28] (see also [26]), refined this condition on the principal symbol to the so-called condition (ψ), and provided strong arguments suggesting that it should be equivalent to local solvability. The necessity of condition (ψ) for local solvability of pseudodifferential equations was proved in two dimensions by R. Moyer in [27] and in general by L. Hörmander ([13]) in 1981. The sufficiency of condition (ψ) for local solvability of differential equations was proved by R. Beals and C. Fefferman ([1]) in 1973; they created a new type of pseudodifferential calculus, based on a Calderón-Zygmund decomposition, and were able to remove the analyticity assumption required by L. Nirenberg and F. Treves. For differential equations in any dimension ([1]) and for pseudodifferential equations in two dimensions ([18], see also [19]), it was shown more precisely that (ψ) implies local solvability with a loss of one derivative with respect to the elliptic case: for a differential operator P of order m (or a pseudodifferential operator in two dimensions), satisfying condition (ψ), $f \in H^s_{\mathrm{loc}}$, the equation $Pu = f$ has a solution $u \in H^{s+m-1}_{\mathrm{loc}}$. In 1994, it was proved by N.L. in [20] (see also [16], [24]) that condition (ψ) does not imply local solvability with loss of one derivative for pseudodifferential equations, contradicting repeated claims by several authors. However in 1996, N. Dencker in [4], proved that these counterexamples were indeed locally solvable, but with a loss of two derivatives.

In [5], N. Dencker claimed that he can prove that condition (ψ) implies local solvability with loss of two derivatives; this preprint contains several breakthrough ideas on the control of the second derivatives subsequent to condition (ψ) and on the choice of the multiplier. The paper [7] contains a proof of local solvability with loss of two derivatives under condition (ψ), providing the final step in the proof of the Nirenberg-Treves conjecture; the more recent paper [6] is providing a proof of local solvability with loss of $\epsilon + \frac{3}{2}$ derivatives under condition (ψ), for any positive ϵ. In the present article, we show that the loss can be limited to $3/2$ derivatives, dropping the ϵ in the previous result. We follow the pattern of Dencker's paper and give some improvements on the key energy estimates.

Acknowledgement. — For several months, I have had the privilege of exchanging several letters and files with Lars Hörmander on the topic of solvability. I am most grateful for the help generously provided. These personal communications are referred to in the text as [17] and are important in all sections of the present paper.

1.2. Statement of the result. — Let P be a properly supported principal-type pseudodifferential operator in a C^∞ manifold $\mathcal{M}$, with principal (complex-valued)[(1)] symbol p. The symbol p is assumed to be a C^∞ homogeneous[(2)] function of degree m on $\dot{T}^*(\mathcal{M})$, the cotangent bundle minus the zero section. The principal type assumption that we shall use here is that

$$(x,\xi) \in \dot{T}^*(\mathcal{M}), \quad p(x,\xi) = 0 \Longrightarrow \partial_\xi p(x,\xi) \neq 0. \tag{1.2.1}$$

Also, the operator P will be assumed of polyhomogeneous type, which means that its total symbol is equivalent to $p + \sum_{j\geq 1} p_{m-j}$, where p_k is a smooth homogeneous function of degree k on $\dot{T}^*(\mathcal{M})$.

Definition 1.2.1 (Condition (ψ)). — Let p be a C^∞ homogeneous function on $\dot{T}^*(\mathcal{M})$. The function p is said to satisfy condition (ψ) if, for $z = 1$ or i, $\operatorname{Im} zp$ does not change sign from $-$ to $+$ along an oriented bicharacteristic of $\operatorname{Re} zp$.

It is a non-trivial fact that condition (ψ) is invariant by multiplication by an complex-valued smooth elliptic factor (see section 26.4 in [14]).

Theorem 1.2.2. — *Let P be as above, such that its principal symbol p satisfies condition (ψ). Let s be a real number. Then, for all $x \in \mathcal{M}$, there exists a neighborhood V such that for all $f \in H^s_{loc}$, there exists $u \in H^{s+m-\frac{3}{2}}_{loc}$ such that*

$$Pu = f \text{ in } V.$$

Proof. — The proof of this theorem will be given at the end of section 4. $\square$

Note that our loss of derivatives is equal to 3/2. The paper [20] proves that solvability with loss of one derivative does *not* follow from condition (ψ), so we have to content ourselves with a loss strictly greater than one. However, the number 3/2 is not likely to play any significant rôle and one should probably expect a loss of $1+\epsilon$ derivatives under condition (ψ). In fact, for the counterexamples given in [20], it seems (but it has not been proven) that there is only a "logarithmic" loss, *i.e.*, the solution u should satisfy $u \in \log\langle D_x\rangle\big(H^{s+m-1}\big)$.

Nevertheless, the methods used in the present article are strictly limited to providing a 3/2 loss. We refer the reader to our appendix A.4 for an argument involving a Hilbertian lemma on a simplified model. This is of course in sharp contrast with operators satisfying condition (P) such as differential operators satisfying condition (ψ). Let us recall that condition (P) is simply ruling out any change of sign of $\operatorname{Im}(zp)$ along the oriented Hamiltonian flow of $\operatorname{Re}(zp)$. Under condition (P) ([1]) or under condition (ψ) in two dimensions ([18]),

(1) Naturally the local solvability of real principal type operators is also a consequence of the next theorem, but much stronger results for real principal type equations were already established in the 1955 paper [9] (see also section 26.1 in [14]).

(2) Here and in the sequel, "homogeneous" will always mean positively homogeneous.

local solvability occurs with a loss of one derivative, the "optimal" loss, and in fact the same as for $\partial/\partial x_1$. One should also note that the semi-global existence theorems of [12] (see also theorem 26.11.2 in [14]) involve a loss of $1+\epsilon$ derivatives. However in that case there is no known counterexample which would ensure that this loss is unavoidable.

Remark 1.2.3. — Theorem 1.2.2 will be proved by a multiplier method, involving the computation of $\langle Pu, Mu\rangle$ with a suitably chosen operator M. It is interesting to notice that, the greater is the loss of derivatives, the more regular should be the multiplier in the energy method. As a matter of fact, the Nirenberg-Treves multiplier of [30] is not even a pseudodifferential operator in the $S^0_{1/2,1/2}$ class, since it could be as singular as the operator $\operatorname{sign} D_{x_1}$; this does not create any difficulty, since the loss of derivatives is only 1. On the other hand, in [4], [23], where estimates with loss of 2 derivatives are handled, the regularity of the multiplier is much better than $S^0_{1/2,1/2}$, since we need to consider it as an operator of order 0 in an asymptotic class defined by an admissible metric on the phase space.

N.B. — For microdifferential operators acting on microfunctions, the sufficiency of condition (ψ) was proven by J.-M. Trépreau [32] (see also [15]), so the present paper is concerned only with the C^∞ category.

1.3. Some notations. — First of all, we recall the definition of the Weyl quantization a^w of a function $a \in \mathcal{S}(\mathbb{R}^{2n})$: for $u \in \mathcal{S}(\mathbb{R}^n)$,

$$(a^w u)(x) = \iint e^{2i\pi(x-y)\xi} a\Big(\frac{x+y}{2}, \xi\Big) u(y)dy. \tag{1.3.1}$$

Our definition of the Fourier transform $\hat{u}$ of $u \in \mathcal{S}(\mathbb{R}^n)$ is $\hat{u}(\xi) = \int e^{-2i\pi x\xi} u(x)dx$ and the usual quantization $a(x, D_x)$ of $a \in \mathcal{S}(\mathbb{R}^{2n})$ is $(a(x, D_x)u)(x) = \int e^{2i\pi x\xi} a(x,\xi)\hat{u}(\xi)d\xi$. The phase space $\mathbb{R}^n_x \times \mathbb{R}^n_\xi$ is a symplectic vector space with the standard symplectic form

$$\big[(x,\xi),(y,\eta)\big] = \langle \xi, y\rangle - \langle \eta, x\rangle. \tag{1.3.2}$$

Definition 1.3.1. — Let g be a metric on $\mathbb{R}^{2n}$, *i.e.*, a mapping $X \mapsto g_X$ from $\mathbb{R}^{2n}$ to the cone of positive definite quadratic forms on $\mathbb{R}^{2n}$. Let M be a positive function defined on $\mathbb{R}^{2n}$.

(1) The metric g is said to be slowly varying whenever $\exists C > 0$, $\exists r > 0$, $\forall X, Y, T \in \mathbb{R}^{2n}$,

$$g_X(Y - X) \le r^2 \Longrightarrow C^{-1} g_Y(T) \le g_X(T) \le C g_Y(T).$$

(2) The symplectic dual metric g^σ is defined as $g^\sigma_X(T) = \sup_{g_X(U)=1}[T,U]^2$. The parameter of g is defined as $\lambda_g(X) = \inf_{T\neq 0}\big(g^\sigma_X(T)/g_X(T)\big)^{1/2}$ and we shall say that g satisfies the uncertainty principle if $\inf_X \lambda_g(X) \ge 1$.

(3) The metric g is said to be temperate when $\exists C > 0$, $\exists N \geq 0$, $\forall X, Y, T \in \mathbb{R}^{2n}$,

$$g_X^\sigma(T) \leq C g_Y^\sigma(T)\big(1 + g_X^\sigma(X-Y)\big)^N.$$

When the three properties above are satisfied, we shall say that g is admissible. The constants appearing in (1) and (3) will be called the structure constants of the metric g.

(4) The function M is said to be g-slowly varying if $\exists C > 0$, $\exists r > 0$, $\forall X, Y \in \mathbb{R}^{2n}$,

$$g_X(Y-X) \leq r^2 \Longrightarrow C^{-1} \leq \frac{M(X)}{M(Y)} \leq C.$$

(5) The function M is said to be g-temperate if $\exists C > 0, \exists N \geq 0$, $\forall X, Y \in \mathbb{R}^{2n}$,

$$\frac{M(X)}{M(Y)} \leq C\big(1 + g_X^\sigma(X-Y)\big)^N.$$

When M satisfies (4) and (5), we shall say that M is a g-weight.

Definition 1.3.2. — Let g be a metric on $\mathbb{R}^{2n}$ and M be a positive function defined on $\mathbb{R}^{2n}$. The set $S(M,g)$ is defined as the set of functions $a \in C^\infty(\mathbb{R}^{2n})$ such that, for all $l \in \mathbb{N}$, $\sup_X \|a^{(l)}(X)\|_{g_X} M(X)^{-1} < \infty$, where $a^{(l)}$ is the l-th derivative. It means that $\forall l \in \mathbb{N}, \exists C_l, \forall X \in \mathbb{R}^{2n}, \forall T_1, \dots, T_l \in \mathbb{R}^{2n}$,

$$|a^{(l)}(X)(T_1, \dots, T_l)| \leq C_l M(X) \prod_{1 \leq j \leq l} g_X(T_j)^{1/2}.$$

Remark. — If g is a slowly varying metric and M is g-slowly varying, there exists $M_* \in S(M,g)$ such that there exists $C > 0$ depending only on the structure constants of g such that

$$\forall X \in \mathbb{R}^{2n}, \quad C^{-1} \leq \frac{M_*(X)}{M(X)} \leq C. \tag{1.3.3}$$

That remark is classical and its proof is sketched in the appendix A.2.

1.4. Partitions of unity.

— We refer the reader to the chapter 18 in [14] for the basic properties of admissible metrics as well as for the following lemma.

Lemma 1.4.1. — *Let g be an admissible metric on $\mathbb{R}^{2n}$. There exists a sequence $(X_k)_{k\in\mathbb{N}}$ of points in the phase space $\mathbb{R}^{2n}$ and positive numbers r_0, N_0, such that the following properties are satisfied. We define U_k, U_k^*, U_k^{**} as the $g_k = g_{X_k}$ balls with center X_k and radius $r_0, 2r_0, 4r_0$. There exist two families of non-negative smooth functions on $\mathbb{R}^{2n}$, $(\chi_k)_{k\in\mathbb{N}}$, $(\psi_k)_{k\in\mathbb{N}}$ such that*

$$\sum_k \chi_k(X) = 1, \ \operatorname{supp}\chi_k \subset U_k, \qquad \psi_k \equiv 1 \ \text{ on } U_k^*, \ \operatorname{supp}\psi_k \subset U_k^{**}.$$