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SEMISTABILITY OF FROBENIUS DIRECT IMAGES OVER CURVES

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ABSTRACT. — Let X be a smooth projective curve of genus $g \geq 2$ defined over an algebraically closed field k of characteristic $p > 0$. Given a semistable vector bundle E over X, we show that its direct image F_*E under the Frobenius map F of X is again semistable. We deduce a numerical characterization of the stable rank-p vector bundles F_*L, where L is a line bundle over X.

RÉSUMÉ (*Semi-stabilité des images directes par Frobenius sur les courbes*)
Soit X une courbe projective lisse de genre ≥ 2 définie sur un corps k algébriquement clos de caractéristique $p > 0$. Étant donné un fibré vectoriel semi-stable E sur X, nous montrons que l'image directe F_*E par le morphisme de Frobenius F de X est aussi semi-stable. Nous déduisons une caractérisation numérique du fibré vectoriel stable F_*L de rang p, où L est un fibré en droites sur X.

1. Introduction

Let X be a smooth projective curve of genus $g \geq 2$ defined over an algebraically closed field k of characteristic $p > 0$ and let $F : X \to X_1$ be the relative k-linear Frobenius map. It is by now a well-established fact that on

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any curve X there exist semistable vector bundles E such that their pull-back under the Frobenius map F^*E is not semistable [4, 5]. In order to control the degree of instability of the bundle F^*E, one is naturally lead (through adjunction $\mathrm{Hom}_{\mathcal{O}_X}(F^*E, E') = \mathrm{Hom}_{\mathcal{O}_{X_1}}(E, F_*E'))$ to ask whether semistability is preserved by direct image under the Frobenius map. The answer is (somewhat surprisingly) yes. In this note we show the following result.

THEOREM 1.1. — *Assume that $g \geq 2$. If E is a semistable vector bundle over X (of any degree), then F_*E is also semistable.*

Unfortunately we do not know whether also stability is preserved by direct image under Frobenius. It has been shown that F_*L is stable for a line bundle L (see [4, Proposition 1.2]) and that in small characteristics the bundle F_*E is stable for any stable bundle E of small rank (see [3]). The main ingredient of the proof is Faltings' cohomological criterion of semistability. We also need the fact that the generalized Verschiebung V, defined as the rational map from the moduli space $\mathcal{M}_{X_1}(r)$ of semistable rank-r vector bundles over X_1 with fixed trivial determinant to the moduli space $\mathcal{M}_X(r)$ induced by pull-back under the relative Frobenius map F,

$$V_r : \mathcal{M}_{X_1}(r) \dashrightarrow \mathcal{M}_X(r), \qquad E \longmapsto F^*E$$

is dominant for large r. We actually show a stronger statement for large r.

PROPOSITION 1.2. — *If $\ell \geq g(p-1)+1$ and ℓ prime, then the generalized Verschiebung V_ℓ is generically étale for any curve X. In particular V_ℓ is separable and dominant.*

As an application of Theorem 1.1 we obtain an upper bound of the rational invariant ν of a vector bundle E, defined as

$$\nu(E) := \mu_{\max}(F^*E) - \mu_{\min}(F^*E),$$

where $\mu_{\max}$ (resp. $\mu_{\min}$) denotes the slope of the first (resp. last) piece in the Harder-Narasimhan filtration of F^*E.

PROPOSITION 1.3. — *For any semistable rank-r vector bundle E*

$$\nu(E) \leq \min\big((r-1)(2g-2), (p-1)(2g-2)\big).$$

We note that the inequality $\nu(E) \leq (r-1)(2g-2)$ was proved in [10, Corollary 2], and in [11, Theorem 3.1]. We suspect that the relationship between both inequalities comes from the conjectural fact that the length (= number of pieces) of the Harder-Narasimhan filtration of F^*E is at most p for semistable E.

Finally we show that direct images of line bundles under Frobenius are characterized by maximality of the invariant ν.

Proposition 1.4. — *Let E be a stable rank-p vector bundle over X. Then the following statements are equivalent.*

1) *There exists a line bundle L such that $E = F_*L$.*
2) $\nu(E) = (p-1)(2g-2)$.

We do not know whether the analogue of this proposition remains true for higher rank.

2. Reduction to the case $\mu(E) = g - 1$

In this section we show that it is enough to prove Theorem 1.1 for semistable vector bundles E with slope $\mu(E) = g - 1$.

Let E be a semistable vector bundle over X of rank r and let s be the integer defined by the equality

$$\mu(E) = g - 1 + \frac{s}{r}\cdot$$

Applying the Grothendieck-Riemann-Roch theorem to the Frobenius map $F : X \to X_1$, we obtain

$$\mu(F_*E) = g - 1 + \frac{s}{pr}\cdot$$

Let $\pi : \widetilde{X} \to X$ be a connected étale covering of degree n and let $\pi_1 : \widetilde{X}_1 \to X_1$ denote its twist by the Frobenius of k (see [9, Section 4]). The diagram

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{F} & \widetilde{X}_1 \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi_1} \\ X & \xrightarrow{F} & X_1 \end{array} \tag{2.1}$$

is Cartesian and we have an isomorphism

$$\pi_1^*(F_*E) \cong F_*(\pi^*E).$$

Since semistability is preserved under pull-back by a separable morphism of curves, we see that π^*E is semistable. Moreover if $F_*(\pi^*E)$ is semistable, then F_*E is also semistable.

Let L be a degree d line bundle over $\widetilde{X}_1$. The projection formula

$$F_*(\pi^*E \otimes F^*L) = F_*(\pi^*E) \otimes L$$

shows that semistability of $F_*(\pi^*E)$ is equivalent to semistability of

$$F_*(\pi^*E \otimes F^*L).$$

Let $\tilde{g}$ denote the genus of $\widetilde{X}$. By the Riemann-Hurwitz formula, one has $\tilde{g}-1=n(g-1)$. We compute

$$\mu(\pi^*E\otimes F^*L)=n(g-1)+n\frac{s}{r}+pd=\tilde{g}-1+n\frac{s}{r}+pd,$$

which gives

$$\mu\big(F_*(\pi^*E\otimes F^*L)\big)=\tilde{g}-1+n\frac{s}{pr}+d.$$

Lemma 2.1. — *For any integer m there exists a connected étale covering $\pi:\widetilde{X}\to X$ of degree $n=p^km$ for some k.*

Proof. — If the p-rank of X is nonzero, the statement is clear. If the p-rank is zero, we know by [9, Corollaire 4.3.4], that there exist connected étale coverings $Y\to X$ of degree p^t for infinitely many integers t (more precisely for all t of the form $(\ell-1)(g-1)$ where ℓ is a large prime). Now we decompose $m=p^su$ with p not dividing u. We then take a covering $Y\to X$ of degree p^t with $t\geq s$ and a covering $\widetilde{X}\to Y$ of degree u. □

Now the lemma applied to the integer $m=pr$ shows existence of a connected étale covering $\pi:\widetilde{X}\to X$ of degree $n=p^km$. Hence $n\frac{s}{pr}$ is an integer and we can take d such that $n\frac{s}{pr}+d=0$.

To summarize, we have shown that for any semistable E over X there exists a covering $\pi:\widetilde{X}\to X$ and a line bundle L over $\widetilde{X}_1$ such that the vector bundle $\widetilde{E}:=\pi^*E\otimes F^*L$ is semistable with $\mu(\widetilde{E})=\tilde{g}-1$ and such that semistability of $F_*\widetilde{E}$ implies semistability of F_*E.

3. Proof of Theorem 1.1

In order to prove semistability of F_*E we shall use the cohomological criterion of semistability due to Faltings [2].

Proposition 3.1 (see [6, Théorème 2.4]). — *Let E be a rank-r vector bundle over X with $\mu(E)=g-1$ and ℓ an integer $>\frac{1}{4}r^2(g-1)$. Then E is semistable if and only if there exists a rank-ℓ vector bundle G with trivial determinant such that*

$$h^0(X,E\otimes G)=h^1(X,E\otimes G)=0.$$

Moreover if the previous condition holds for one bundle G, it holds for a general bundle by upper semicontinuity of the function $G\mapsto h^0(X,E\otimes G)$.

Remark. — The proof of this proposition (see [6, Section 2.4]) works over any algebraically closed field k.