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ON A CERTAIN GENERALIZATION OF SPHERICAL TWISTS

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ABSTRACT. — This note gives a generalization of spherical twists, and describe the autoequivalences associated to certain non-spherical objects. Typically these are obtained by deforming the structure sheaves of $(0,-2)$-curves on threefolds, or deforming $\mathbb{P}$-objects introduced by D. Huybrechts and R. Thomas.

RÉSUMÉ (*Sur une généralisation des twists sphériques*). — Cette note donne une généralisation des twists sphériques et décrit des auto-équivalences associées à certains objets qui ne sont pas sphériques. Typiquement ces objets sont obtenus par déformation du faisceau structural d'une $(0,2)$-courbe dans une variété de dimension trois ou d'un $\mathbb{P}$-objet introduit par D. Huybrechts et R. Thomas.

1. Introduction

We introduce a new class of autoequivalences of derived categories of coherent sheaves on smooth projective varieties, which generalizes the notion of spherical twists given in [12]. Such autoequivalences are associated to a certain class of objects, which are not necessary spherical but are interpreted as "fat"

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version of them. We introduce the notion of R-spherical objects for a noetherian and artinian local $\mathbb{C}$-algebra R, and imitate the construction of spherical twists to give the associated autoequivalences.

Let X be a smooth complex projective variety, and $D(X)$ be a bounded derived category of coherent sheaves on X. When X is a Calabi-Yau 3-fold, $D(X)$ is considered to represent the category of D-branes of type B, and should be equivalent to the derived Fukaya category on a mirror manifold under Homological mirror symmetry [8]. On the mirror side, there are typical symplectic automorphisms by taking Dehn twists along Lagrangian spheres [11]. The notions of spherical objects and associated twists were introduced in [12] in order to realize Dehn twists under mirror symmetry. Recall that $E \in D(X)$ is called *spherical* if the following holds [12]:

- $\operatorname{Ext}^i_X(E,E) = \begin{cases} \mathbb{C} & \text{if } i=0 \text{ or } i=\dim X, \\ 0 & \text{otherwise;} \end{cases}$
- $E \otimes \omega_X \cong E$.

Then one can construct the autoequivalence $T_E\colon D(X) \to D(X)$ which fits into the distinguished triangle [12]:

$$\mathbb{R}\operatorname{Hom}(E,F) \otimes_{\mathbb{C}} E \longrightarrow F \longrightarrow T_E(F),$$

for $F \in D(X)$. The autoequivalence T_E is called a *spherical twist*. This is a particularly important class of autoequivalences, especially when we consider A_n-configulations on surfaces as indicated in [7]. On the other hand, it has been observed that there are some autoequivalences which are not described in terms of spherical twists. This occurs even in the similar situation discussed in [7] as follows. Let $X \to Y$ be a three dimensional flopping contraction which contracts a rational curve $C \subset X$, and $X^\dagger \to Y$ be its flop. Then one can construct the autoequivalence [1, 3, 4],

$$\Phi := \Phi^{\mathcal{O}_{X \times_Y X^\dagger}}_{X^\dagger \to X} \circ \Phi^{\mathcal{O}_{X \times_Y X^\dagger}}_{X \to X^\dagger} : D(X) \longrightarrow D(X^\dagger) \longrightarrow D(X).$$

If $C \subset X$ is not a $(-1,-1)$-curve, Φ is not written as a spherical twist, and our motivation comes from describing such autoequivalences. Let R be a noetherian and artinian local $\mathbb{C}$-algebra. We introduce the notion of R-spherical objects defined on $D(X \times \operatorname{Spec} R)$. In the above example, $\operatorname{Spec} R$ is taken to be the moduli space of $\mathcal{O}_C(-1)$, and the universal family gives the R-spherical object. Our main theorem is the following:

THEOREM 1.1. — *To any R-spherical object $\mathcal{E} \in D(X \times \operatorname{Spec} R)$, we can associate the autoequivalence $T_{\mathcal{E}}\colon D(X) \to D(X)$, which fits into the distinguished triangle*

$$\mathbb{R}\operatorname{Hom}_X(\pi_*\mathcal{E}, F) \overset{\mathbb{L}}{\otimes}_R \pi_*\mathcal{E} \longrightarrow F \longrightarrow T_{\mathcal{E}}(F),$$

for $F \in D(X)$. *Here* $\pi\colon X \times \operatorname{Spec} R \to X$ *is the projection.*

Using the notion of R-spherical objects and associated twists, we can also give the deformations of $\mathbb{P}$-twists in the case which is not treated in [5].

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Notations and conventions

- For a variety X, we denote by $D(X)$ its bounded derived category of coherent sheaves.
- Δ means the diagonal $\Delta \subset X \times X$ or the diagonal embedding $\Delta\colon X \to X \times X$.
- For another variety Y and an object $\mathcal{P} \in D(X \times Y)$, denote by $\Phi^{\mathcal{P}}_{X\to Y}$ the integral transform with kernel $\mathcal{P}$, *i.e.*,
$$\Phi^{\mathcal{P}}_{X\to Y}(*) := \mathbb{R}p_{Y*}(p_X^*(*) \overset{\mathbb{L}}{\otimes} \mathcal{P})\colon D(X) \longrightarrow D(Y).$$
Here p_X, p_Y are projections from $X \times Y$ onto corresponding factors.

2. Generalized spherical twists

Let X be a smooth projective variety over $\mathbb{C}$ and R be a noetherian and artinian local $\mathbb{C}$-algebra. We introduce the notion of R-spherical objects defined on $D(X \times \operatorname{Spec} R)$. Let $\pi\colon X \times \operatorname{Spec} R \to X$ and $\pi'\colon X \times \operatorname{Spec} R \to \operatorname{Spec} R$ be projections and $0 \in \operatorname{Spec} R$ be the closed point.

Definition 2.1. — An object $\mathcal{E} \in D(X \times \operatorname{Spec} R)$ is called *R-spherical* if the following conditions hold:

- $\mathcal{E}$ is represented by a bounded complex $\mathcal{E}^\bullet$ with each $\mathcal{E}^i$ a coherent $\mathcal{O}_{X\times\operatorname{Spec} R}$-module flat over R. In particular we have the bounded derived restriction $E := \mathcal{E}^\bullet|_{X\times\{0\}} \in D(X)$.
- $\operatorname{Ext}^i_X(E,E) = \begin{cases} \mathbb{C} & \text{if } i = 0 \text{ or } i = \dim X, \\ 0 & \text{otherwise;} \end{cases}$
- $E \otimes \omega_X \cong E$.

Remark 2.2. — If $R = \mathbb{C}$, then R-spherical objects coincide with usual spherical objects.

We imitate the construction of the spherical twists in the following theorem.

THEOREM 2.3. — *To any R-spherical object $\mathcal{E} \in D(X \times \operatorname{Spec} R)$, we can associate the autoequivalence $T_{\mathcal{E}} \colon D(X) \to D(X)$, which fits into the distinguished triangle:*

$$\mathbb{R}\operatorname{Hom}_X(\pi_*\mathcal{E}, F) \overset{\mathbb{L}}{\otimes}_R \pi_*\mathcal{E} \longrightarrow F \longrightarrow T_{\mathcal{E}}(F),$$

for $F \in D(X)$. Here R-module structures on $\mathbb{R}\operatorname{Hom}_X(\pi_*\mathcal{E}, F)$ and $\pi_*\mathcal{E}$ are inherited from R-module structure on $\mathcal{E}$.

Proof. — First we construct the kernel of $T_{\mathcal{E}}$. Let p_{ij} and p_i be projections as in the following diagram

$$\begin{tikzcd}
 & X \times \operatorname{Spec} R \times X \arrow[dl, "p_{12}"'] \arrow[d, "p_{13}"] \arrow[dr, "p_{23}"] & \\
X \times \operatorname{Spec} R \arrow[d, "\pi"] & X \times X \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & X \times \operatorname{Spec} R, \arrow[d, "\pi"] \\
X & & X
\end{tikzcd}$$

and consider the object

$$\mathcal{Q} := \mathbb{R}p_{13*}\big(p_{12}^*(\pi^!\mathcal{O}_X \overset{\mathbb{L}}{\otimes} \check{\mathcal{E}}) \overset{\mathbb{L}}{\otimes} p_{23}^*\mathcal{E}\big) \in D(X \times X).$$

Here $\check{\mathcal{E}}$ means its derived dual. Then for $F \in D(X)$, we can calculate $\Phi_{X\to X}^{\mathcal{Q}}(F)$ as follows:

$$\begin{aligned}
\Phi_{X\to X}^{\mathcal{Q}}(F) &\cong \mathbb{R}p_{2*}(\mathbb{R}p_{13*}(p_{12}^*(\pi^!\mathcal{O}_X \overset{\mathbb{L}}{\otimes} \check{\mathcal{E}}) \overset{\mathbb{L}}{\otimes} p_{23}^*\mathcal{E}) \overset{\mathbb{L}}{\otimes} p_1^*F) \\
&\cong \mathbb{R}p_{2*}\mathbb{R}p_{13*}(p_{12}^*(\pi^!\mathcal{O}_X \overset{\mathbb{L}}{\otimes} \check{\mathcal{E}}) \overset{\mathbb{L}}{\otimes} p_{23}^*\mathcal{E} \overset{\mathbb{L}}{\otimes} p_{13}^*p_1^*F) \\
&\cong \pi_*\mathbb{R}p_{23*}(p_{12}^*(\pi^!\mathcal{O}_X \overset{\mathbb{L}}{\otimes} \check{\mathcal{E}}) \overset{\mathbb{L}}{\otimes} p_{23}^*\mathcal{E} \overset{\mathbb{L}}{\otimes} p_{12}^*\pi^*F) \\
&\cong \pi_*\big\{\mathcal{E} \overset{\mathbb{L}}{\otimes} \mathbb{R}p_{23*}p_{12}^*(\pi^!\mathcal{O}_X \overset{\mathbb{L}}{\otimes} \check{\mathcal{E}} \overset{\mathbb{L}}{\otimes} \pi^*F)\big\} \\
&\cong \pi_*\big\{\mathcal{E} \overset{\mathbb{L}}{\otimes} \pi'^*\mathbb{R}\pi'_*\mathbb{R}\mathcal{H}om(\mathcal{E}, \pi^!F)\big\} \\
&\cong \pi_*\mathcal{E} \overset{\mathbb{L}}{\otimes}_R \mathbb{R}\operatorname{Hom}(\pi_*\mathcal{E}, F).
\end{aligned}$$

The fifth equality comes from the base change formula for the diagram below:

$$\begin{tikzcd}
X \times \operatorname{Spec} R \times X \arrow[r, "p_{12}"] \arrow[d, "p_{23}"'] & X \times \operatorname{Spec} R \arrow[d, "\pi'"] \\
X \times \operatorname{Spec} R \arrow[r, "\pi'"] & \operatorname{Spec} R.
\end{tikzcd}$$