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ON BRODY AND ENTIRE CURVES

BY JÖRG WINKELMANN

ABSTRACT. — We discuss an example of an open subset of a torus which admits a dense entire curve, but no dense Brody curve.

RÉSUMÉ (*Sur les courbes de Brody et les courbes entières*). — On présente un exemple de sous-ensemble de tore qui possède une courbe entière dense mais pas de courbe de Brody.

1. Introduction

1.1. Brody's theorem. — Let Y be a complex manifold. It is called "taut" if the family of all holomorphic maps $f : \Delta \to Y$ is a normal family. Let us from now on assume that Y is compact. Then being "taut" is easily seen to be equivalent with hyperbolicity in the sense of Kobayashi. The theorem of Brody (see [3]) states that this is furthermore equivalent with the property that every holomorphic map from $\mathbb{C}$ to Y is constant. (Remark: This can be regarded as a special case of a heuristic philosophy known as "Bloch's principle", see [12].)

Now we may raise the question: What about holomorphic maps to a compact complex manifold fixing some given base points? Given a compact complex manifold Y and a point $y \in Y$, let us consider the following two statements:

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- *Every holomorphic map $f : \mathbb{C} \to Y$ with $f(0) = y$ is constant.*
- *The family of all holomorphic maps $f : \Delta \to Y$ with $f(0) = y$ is a normal family.*

Are they equivalent?

Using the notion of the infinitesimal Kobayashi-Royden pseudometric as introduced in [10] this can be reformulated into the following question: *"If the infinitesimal Kobayashi-Royden pseudometric on a compact complex manifold Y degenerates for some point $y \in Y$, does this imply that there exists a holomorphic map $f : \mathbb{C} \to Y$ with $y \in f(\mathbb{C})$?"*

Thanks to Brody's theorem it is clear that there exists some non-constant holomorphic map $f : \mathbb{C} \to Y$ if the Kobayashi-Royden pseudometric is degenerate at some point y of Y. But it is not clear that f can be chosen in such a way that y is in the image or at least in the closure of the image. Of course, at first it looks absurd that degeneracy of the Kobayashi-Royden pseudometric at one point y should only imply the existence of a non-constant holomorphic map to some part of Y far away of y and should not imply the existence of a non-constant map $f : \mathbb{C} \to Y$ whose image comes close to y.

Thus one is led to ask:

Question 1. — *Let X be a compact complex manifold, $x \in X$. Assume that the infinitesimal Kobayashi-Royden pseudometric is degenerate on T_xX. Does this imply that there exists a non-constant holomorphic map $f : \mathbb{C} \to X$ with $f(0) = x$?*

Since Brody parametrization always yields a holomorphic map $f : \mathbb{C} \to X$ whose derivative is bounded, one might be inclined to even ask:

Question 2. — *Let X be a compact complex manifold, $x \in X$. Assume that the infinitesimal Kobayashi-Royden pseudometric is degenerate on T_xX. Does this imply that there exists a non-constant holomorphic map $f : \mathbb{C} \to X$ with $f(0) = x$ such that the derivative $f' : \mathbb{C} \to TX$ is bounded (with respect to the euclidean metric on $\mathbb{C}$ and some hermitian metric on X)?*

In this article we give examples which show that the answer to the second question is negative. The answer to the first question remains open.

1.2. Reparametrization. — The key idea for proving Brody's theorem is the following: Let $f_n : \Delta \to Y$ be a non-normal family. Then we look for an increasing sequence of disk Δ_{r_n} which exhausts $\mathbb{C}$ (i.e. $\lim r_n = +\infty$) and a sequence of holomorphic maps $\alpha_n : \Delta_{r_n} \to \Delta$ such that a subsequence of $f_n \circ \alpha_n$ converges (locally uniformly) to a non-constant holomorphic map from $\mathbb{C}$ to Y.

In his proof Brody used combinations of affine-linear maps with automorphisms of the disk for the α_n.

Zalcman [12] investigated other reparametrizations where the α_n themselves are affine-linear maps, a concept which has the advantage that it can also be applied to harmonic maps.

1.3. Subvarieties of abelian varieties. — Let A be a complex abelian variety (i.e. a compact complex torus which is simultaneously a projective algebraic variety) and X a subvariety. Let E denote the union of all translates of complex subtori of A which are contained in X. It is known that this union is either all of X or a proper algebraic subvariety (see [6]).

Since A is a compact complex torus there is a flat hermitian metric on A induced by the euclidean metric on $\mathbb{C}^g$ via $A \simeq \mathbb{C}^g/\Gamma$. A holomorphic map $f : \mathbb{C} \to A$ has bounded derivative with respect to this metric if and only if it is induced by an affine-linear map from $\mathbb{C}$ to $\mathbb{C}^g$.

From this, one can deduce that $f(\mathbb{C}) \subset E$ for every holomorphic map $f : \mathbb{C} \to X$ with bounded derivative. In fact, $f(\mathbb{C}) \subset E$ holds for very holomorphic map $f : \mathbb{C} \to X$; this is a consequence of the theorem of Bloch-Ochiai.

It is thus natural to conjecture:

CONJECTURE. — *The Kobayashi-pseudodistance on X is a distance outside E.*

For example, this statement is a consequence of the more general conjecture VIII.I.4 by S. Lang [9]. In the context of classification theory the above statement has also been conjectured by F. Campana [4, §9.3].

In the spirit of the analogue between diophantine geometry and entire holomorphic curves as pointed out by Vojta [11], the conjecture above is also supported by the famous result of Faltings [5] with which he solved the Mordell conjecture. This result states the following: If we assume that A and X are defined over a number field K, then with only finitely many exceptions every K-rational point of X is contained in E.

1.4. The first example. — We construct an example of the following type: There is an abelian surface T with open subsets $\Omega_2 \subset \Omega_1 \subset T$ such that there exists an entire curve $f : \mathbb{C} \to \Omega_1$ for which the image is dense in Ω_1, but for every non-constant Brody curve $f : \mathbb{C} \to T$ whose image is contained in Ω_1 the closure of the image is a compact complex curve inside of Ω_2 (and Ω_2 is not dense in Ω_1).

1.5. The second example. — We show that by blowing up a suitably chosen curve on a suitably chosen three-dimensional abelian variety, one can obtain a compact complex manifold X with a hypersurface Z such that Z contains the image of every non-constant Brody curve, although X does admit an entire curve with dense image and the infinitesimal Kobayashi-Royden pseudometric vanishes identically on X.

1.6. Why two examples?— Although the second example suffices to show that the answer to the second question is negative (and thus to show that the behaviour of Brody curves is fundamentally different from that of arbitrary entire curves), we include a description of the first example (which was obtained earlier), because we feel that it is of independent interest. The methods for constructing the two examples are completely different, and the first example might also be interesting from the point of view of studying entire curves in compact complex tori.

For example, for every entire curve with values in a compact complex torus the Zariski closure of the image is a translated subtorus, but our example shows that the ordinary closure of the image can be far away from being a translated real subtorus.

2. Some basic facts on Brody curves

We recall some basic facts on Brody curves.

Let X be a complex manifold endowed with some hermitian metric. Then an *entire curve* is a non-constant holomorphic map from $\mathbb{C}$ to X and a *Brody curve* is a non-constant holomorphic map $f : \mathbb{C} \to X$ for which the derivative f' is bounded (with respect to the euclidean metric on $\mathbb{C}$ and the given hermitian metric on X).

- If X is compact, the notion of a "Brody curve" is independent of the choice of the hermitian metric.
- If $\phi : X \to Y$ is a holomorphic map between compact complex manifolds and $f : \mathbb{C} \to X$ is a Brody curve, then $\phi \circ f : \mathbb{C} \to Y$ is a Brody curve, too. (But not necessarily conversely.)