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TOPOLOGICAL DISJOINTNESS FROM ENTROPY ZERO SYSTEMS

by Wen Huang, Kyewon Koh Park & Xiangdong Ye

Abstract. — The properties of topological dynamical systems (X,T) which are disjoint from all minimal systems of zero entropy, $\mathcal{M}_0$, are investigated. Unlike the measurable case, it is known that topological K-systems make up a proper subset of the systems which are disjoint from $\mathcal{M}_0$. We show that (X,T) has an invariant measure with full support, and if in addition (X,T) is transitive, then (X,T) is weakly mixing. A transitive diagonal system with only one minimal point is constructed. As a consequence, there exists a thickly syndetic subset of $\mathbb{Z}_+$, which contains a subset of $\mathbb{Z}_+$ arising from a positive entropy minimal system, but does not contain any subset of $\mathbb{Z}_+$ arising from a zero entropy minimal system. Moreover we study the properties of topological dynamical systems (X,T) which are disjoint from larger classes of zero entropy systems.

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Résumé (*Disjonction topologique des systèmes d'entropie nulle*)
Nous étudions les propriétés des systèmes topologiques dynamiques (X,T) qui sont disjoints de tout système minimal d'entropie nulle $\mathcal{M}_0$. Contrairement au cas mesurable, il est connu que les K-systèmes topologiques constituent un sous-ensemble propre des systèmes disjoints de $\mathcal{M}_0$. Nous montrons que (X,T) a une mesure invariante à support plein, et que si, de plus, (X,T) est transitif alors il est faiblement mélangeant. Nous construisons un système diagonal transitif avec un seul point minimal. Par conséquent, il existe un sous-ensemble grassement syndétique de $\mathbb{Z}_+$, qui contient un sous-ensemble de $\mathbb{Z}_+$, provenant d'un système minimal d'entropie positive, mais qui ne contienne aucun sous-ensemble de $\mathbb{Z}_+$ provenant d'un système minimal d'entropie nulle. D'autre part, nous étudions les propriétés des systèmes topologiques dynamiques (X,T) qui sont disjoints de classes plus larges de systèmes à entropie nulle.

1. Introduction

By a *topological dynamical system* (TDS for short) (X,T), we mean a compact metric space X with a continuous surjective map T from X to itself. A measurable system is defined to be *K-mixing* or a *K-system* if and only if every nontrivial partition has positive entropy. Equivalently a system is K-mixing if and only if it has no nontrivial entropy zero factor. It is well known that if a system is K-mixing, then it is strongly mixing. Entropy pairs, uniformly positive entropy systems (u.p.e.) and completely positive entropy systems (c.p.e.) have been introduced by Blanchard [1], [2] in search for the topological definition of a K-system. We define a system is *topologically* K if and only if every nontrivial finite open cover (each element is not dense) has positive entropy. It is known [11] that if a system is topologically K and minimal, then it is topologically strongly mixing (it is still an open question that minimality and u.p.e. imply strongly mixing property).

In [5] the notion of disjointness was defined both in topological and measure-theoretical settings. If (X,T) and (Y,S) are two TDS we say $J \subset X \times Y$ is a *joining* of X and Y if J is a non-empty closed invariant set and is projected onto X and Y respectively. If there is only one obvious joining $J = X \times Y$, we then say that (X,T) and (Y,S) are *disjoint* or $(X,T) \perp (Y,S)$. Let $\mathcal{M}$ be the collection of all minimal systems. As the disjointness of two TDS implies one of them is minimal, thus this naturally leads us to a question to characterize $\mathcal{M}^\perp$ (Question E in [5]). It is known [5] that a weakly mixing system with a dense set of periodic points is in $\mathcal{M}^\perp$. Recently, the authors [13] have shown that there are transitive systems in $\mathcal{M}^\perp$ with no periodic points. Moreover, it is proved that if $(X,T) \in \mathcal{M}^\perp$ then (X,T) must have a dense set of minimal points, and if in addition (X,T) is transitive it must be weakly mixing.

In ergodic theory, a measurable dynamical system (MDS, for short) is a K-system if and only if it is disjoint from any MDS of zero entropy. One of the motivations of the paper is to see if the above result can be extended to TDS. Note that if two non-trivial TDS are disjoint then one of them is minimal [5] and the other one has a dense set of recurrent points [13]. Hence when we consider disjointness from zero entropy systems, we assume the class of zero entropy systems is either minimal or transitive. In this paper we consider both cases.

When the class consists of minimal zero entropy systems (denoted by $\mathcal{M}_0$), we prove that if $(X,T) \perp \mathcal{M}_0$, then (X,T) has an invariant measure with full support, and if in addition (X,T) is transitive, (X,T) is weakly mixing. Note that it has been proved that transitive diagonal systems which include u.p.e. systems are in $\mathcal{M}_0^{\perp}$ [2]. When the class consists of transitive zero entropy systems with invariant measures of full support (denoted by E_0), we prove that if $(X,T) \perp E_0$, then (X,T) is minimal and has c.p.e. We remark that systems which are disjoint from all transitive zero entropy systems are the trivial ones. If we restrict ourselves to a subclass M_0 (transitive zero entropy TDS with a dense set of minimal points) of E_0, we can show that all minimal diagonal systems are in $M_0^{\perp}$. It is an open question if there is a TDS in $M_0^{\perp} \setminus E_0^{\perp}$.

We construct a transitive diagonal system with only one minimal point. One consequence is that there is a TDS in $\mathcal{M}_0^{\perp} \setminus \mathcal{M}^{\perp}$. Another consequence is that there exists a thickly syndetic subset of $\mathbb{Z}_+$, which contains a subset of $\mathbb{Z}_+$ arising from a positive entropy minimal system, but does not contain any subset of $\mathbb{Z}_+$ arising from a zero entropy minimal system.

Now we recall some of the definitions.

— We say that (X,T) is *transitive* if for each pair of non-empty open subsets U and V, $N(U,V) = \{n \in \mathbb{Z}_+ : T^{-n}V \cap U \neq \varnothing\}$ is infinite, where $\mathbb{Z}_+ = \{0\} \cup \mathbb{N}$.

— (X,T) is *weakly mixing* if $(X \times X, T \times T)$ is transitive.

— We say that $x \in X$ is a *transitive point* if its orbit $orb(x,T) = \{x, T(x), \ldots\}$ is dense in X. It is well known that if (X,T) is transitive then the set of transitive points is a dense G_δ set (denoted by Tran_T) and if $\mathrm{Tran}_T = X$ we say that (X,T) is *minimal*.

— For a TDS (X,T), we call $x \in X$ a *minimal point* if $(\mathrm{cl}(\mathrm{orb}(x,T)),T)$ is a minimal subsystem of (X,T). It is well known that x is a minimal point of (X,T) if and only if for any neighborhood U of x,

$$N(x,U) := \big\{n \in \mathbb{Z}_+ :\ T^n x \in U\big\}$$

is *syndetic*, i.e. with bounded gaps.

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2. TDS with an invariant measure of full support

In this section, we study the properties of (X,T) which is disjoint from the class E_0. Recall that a TDS (X,T) is an *E-system* if it is transitive and has an invariant measure μ with full support, i.e., $\mathrm{supp}(\mu) = X$; it is an *M-system* if it is transitive and the set of minimal points is dense; and it is *topologically ergodic* (TE, for short) if (X,T) is transitive and for each pair of non-empty open subsets U, V of X, $N(U,V)$ is *syndetic*. It is known that a minimal system is an E-system, and an E-system is TE [9]. However there exist a TE-system which is not an E-system, and an E-system which is not minimal. Two TDS are *weakly disjoint* if their product is transitive. Note that if both (X,T) and (Y,S) are transitive and $(X,T) \perp (Y,S)$, then they are weakly disjoint.

Since we need some of the results in [13], we summarize them in the following proposition.

PROPOSITION 2.1. — *Let (X,T) be a TDS.*

1. *If $(X,T) \perp (Y,S)$ with (Y,S) minimal and non-trivial, then $R(T)$ is a dense G_δ set of X, where*
$$R(T) = \{x \in X : \text{ there is } n_i \to +\infty \text{ with } T^{n_i}x \to x\}$$
is the set of all recurrent points of (X,T).
2. *Let T be invertible, V be a non-empty open set of X and*
$$Y = \mathrm{cl}\left(\bigcup_{i\in\mathbb{Z}} T^i V\right).$$
If $(X,T) \perp \mathcal{M}$, then $(Y,T) \perp \mathcal{M}$.
3. *If there are transitive sub-systems (X_i,T) of (X,T) satisfying that $\cup_i X_i$ is dense in X and $(X_i,T) \perp \mathcal{M}$ for each $i \in \mathbb{N}$. Then $(X,T) \perp \mathcal{M}$.*
4. *Let (X,T) be an equicontinuous system. If $(X,T) \in \mathcal{M}^\perp$ then each point of X is a fixed point. Consequently, if (Y,S) is TDS with $(Y,S) \in \mathcal{M}^\perp$, then the maximal equicontinuous factor of (Y,S) consists of fixed points.*
5. *If $(X,T) \in \mathcal{M}^\perp$, then (X,T) has dense minimal points. If in addition, (X,T) is transitive, then (X,T) is weakly mixing.*

According to Proposition 2.1 1), when considering our question of disjointness from entropy zero systems, it is natural to restrict our attention to TDS with dense sets of recurrent points and having zero entropy. The following proposition tells us that this class is too large. Before we go on to prove the proposition, let us recall some more definitions. Let S be a subset of $\mathbb{Z}_+$. The *upper Banach density of S* is
$$BD^*(S) = \limsup_{|I|\to+\infty} \frac{|S \cap I|}{|I|},$$