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LAGRANGIAN FIBRATIONS ON GENERALIZED KUMMER VARIETIES

BY MARTIN G. GULBRANDSEN

ABSTRACT. — We investigate the existence of Lagrangian fibrations on the generalized Kummer varieties of Beauville. For a principally polarized abelian surface A of Picard number one we find the following: The Kummer variety K^nA is birationally equivalent to another irreducible symplectic variety admitting a Lagrangian fibration, if and only if n is a perfect square. And this is the case if and only if K^nA carries a divisor with vanishing Beauville-Bogomolov square.

RÉSUMÉ (*Fibrations lagrangiennes sur les variétés de Kummer généralisées*)

Nous étudions l'existence de fibrations lagrangiennes sur les variétés de Kummer généralisées ăde Beauville. Pour une surface abélienne principalement polarisée dont le nombre de Picard égale 1 nous prouvons le résultat suivant : la variété de Kummer K^nA est birationnellement équivalente à une ăvariété symplectique irréductible admettant une fibration lagrangienne si et seulement si n est un carré parfait. Et cela est le cas si et seulement si K^nA supporte un diviseur dont le carré de Beauville-Bogomolov s'annule.

1. Introduction

Let X denote a projective irreducible symplectic variety of dimension $2n$. We refer the reader to Huybrechts' exposition [6] for definitions and general

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background material. Matsushita [8], [9], [10], [11] studied *fibrations* of X, that is, proper maps

$$f\colon X \longrightarrow B, \tag{1}$$

such that a generic fibre is connected and has positive dimension. Assuming B to be projective and nonsingular, Matsushita showed that every component of every fibre of f is a Lagrangian subvariety of X of dimension n, and every nonsingular fibre is an abelian variety. Furthermore, the base B is n-dimensional Fano and its Hodge numbers agree with those of $\mathbb{P}^n$. It is a conjecture that B is in fact isomorphic to $\mathbb{P}^n$.

The setup can be generalized slightly:

Definition 1.1. — With X and B as above, a rational map

$$f\colon X \dashrightarrow B$$

is a *rational fibration* of X over B if there exist another projective irreducible symplectic variety X' and a birational equivalence $\phi\colon X' \overset{\sim}{\dashrightarrow} X$ such that the composition $f \circ \phi$ is a (regular) fibration of X' over B.

A basic tool in the study of irreducible symplectic varieties is the *Beauville-Bogomolov form*, which is an integral quadratic form q on $H^2(X,\mathbb{Z})$, satisfying

$$q(\alpha)^n = c\deg(\alpha^{2n})$$

for a positive real constant c. A birational map between irreducible symplectic varieties induces an isomorphism on $H^2(-,\mathbb{Z})$, compatible with the Beauville-Bogomolov forms. It follows that in the situation of Definition 1.1, the pullback $D = f^*H$ of any divisor H on B satisfies $q(D) = 0$. Conversely, one may ask:

Question 1.2. — *Suppose X carries a nontrivial divisor D with vanishing Beauville-Bogomolov square. Does X admit a rational fibration over $\mathbb{P}^n$?*

One may try to answer the question for the known examples of projective irreducible symplectic varieties. There are two standard series of examples, both due to Beauville [1]: The first is the Hilbert scheme $S^{[n]}$ (of dimension $2n$) parametrizing finite subschemes of length n of a K3 surface S. The second is the (generalized) *Kummer variety* K^nA (of dimension $2n-2$) associated to an abelian surface A, defined as the fibre of the map

$$\sigma\colon A^{[n]} \longrightarrow A \tag{2}$$

induced by the group law on A. The map σ is locally trivial in the étale topology, and in particular all fibres are isomorphic. So there is no ambiguity in this definition. Recently, Sawon [15] and Markushevich [7] answered Question 1.2 in the affirmative for the Hilbert scheme $S^{[n]}$ of a generic K3 surface. In this text, we consider the case of the Kummer varieties.

To state our result, we need the notion of the dual divisor class: If $C \in \mathrm{Pic}(A)$ is an ample divisor class, then there is a canonically defined dual divisor class $\widehat{C} \in \mathrm{Pic}(\widehat{A})$, which is also ample, and the two divisors C and $\widehat{C}$ have the same self intersections. A precise definition is given in Example 2.4. With this notation, our result is the following:

THEOREM 1.3. — *Let A be an abelian surface carrying an effective divisor $C \subset A$ with self intersection $2n$, where $n > 2$, and assume there exist nonsingular irreducible curves in the linear system $|\widehat{C}|$ on $\widehat{A}$. Then the Kummer variety $K^n A$ admits a rational fibration*

$$f\colon K^n A \dashrightarrow |\widehat{C}| \cong \mathbb{P}^{n-1}. \tag{3}$$

REMARK 1.4. — The assumption that $|\widehat{C}|$ contains nonsingular irreducible curves is only used to verify that a generic fibre of f is connected. We will see in Example 2.4 that this assumption is satisfied whenever A is indecomposable, i.e. not a product of elliptic curves, and also whenever C is nonprimitive, i.e. divisible in the Néron-Severi group of A.

The theorem is proved in Section 3. We have the following corollary, which answers Question 1.2 in the affirmative for the Kummer varieties associated to a generic principally polarized abelian surface, and which is proved in Section 3.5:

COROLLARY 1.5. — *If the abelian surface A has Picard number one and admits a principal polarization, then the following are equivalent, for each $n > 2$:*

1) *The Kummer variety $K^n A$ admits a rational fibration over $\mathbb{P}^{n-1}$.*
2) *$K^n A$ carries a divisor with vanishing Beauville-Bogomolov square.*
3) *n is a perfect square.*

The present work has been carried out independently of the works of Sawon and Markushevich, but the construction is similar. However, Sawon and Markushevich are able to answer Question 1.2 for the Hilbert scheme of *any* (generic) K3 surface, and their construction involves a certain moduli space of twisted sheaves. In this text, we avoid twisted sheaves, but are only able to answer Question 1.2 for (generic) *principally* polarized abelian surfaces. It might be possible to extend the construction to arbitrary polarizations by adapting the use of twisted sheaves in the construction of Sawon and Markushevich. [1]

[1] After this paper was written, K. Yoshioka (arXiv:math.AG/0605190) answered Question 1.2 affirmatively for Kummer varieties of arbitrarily polarized abelian surfaces. The proof uses twisted sheaves.

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2. Preparation

We work in the category of noetherian schemes over $\mathbb{C}$. By a map of schemes we mean a morphism in this category. By a sheaf on a scheme X we mean a coherent $\mathcal{O}_X$-module.

If A is an abelian variety, we denote the identity element for the group law on A by 0, and if a is a point on A, we write $T_a\colon A \to A$ for translation by a. We write $\widehat{A}$ for the dual abelian variety. We denote by $\mathcal{P}_x$ the homogeneous line bundle on A corresponding to a point $x \in \widehat{A}$. If D is a divisor on A, we denote by

$$\phi_D\colon A \longrightarrow \widehat{A}$$

the map that takes a point $a \in A$ to the (invertible sheaf associated to the) divisor $T_a^*D - D$.

We use the same symbol to denote a divisor on a variety, its class in the Picard group and its class in the second cohomology group.

In this section, we recall a few results from the literature on sheaves on abelian surfaces.

2.1. The Fourier-Mukai transform. — Let $X \to T$ be an abelian scheme over T, and let $\widehat{X} \to T$ denote its dual abelian scheme. Let $\mathcal{P}$ be the Poincaré line bundle on $X \times_T \widehat{X}$, normalized such that the restrictions of $\mathcal{P}$ to $X \times 0$ and $0 \times \widehat{X}$ are trivial. Let

$$X \xleftarrow{p} X \times_T \widehat{X} \xrightarrow{q} \widehat{X}$$

denote the two projections.

Following Mukai [12], [13], we define a functor $\widehat{S}$ from the category of $\mathcal{O}_X$-modules to the category of $\mathcal{O}_{\widehat{X}}$-modules by

$$\widehat{S}(\mathcal{E}) = q_*\big(p^*(\mathcal{E}) \otimes \mathcal{P}\big).$$

Reversing the roles of X an $\widehat{X}$, we get a functor S taking an $\mathcal{O}_{\widehat{X}}$-module $\mathcal{F}$ to the $\mathcal{O}_X$-module

$$S(\mathcal{F}) = p_*\big(q^*(\mathcal{F}) \otimes \mathcal{P}\big).$$