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VOLUME OF SPHERES IN DOUBLING METRIC MEASURED SPACES AND IN GROUPS OF POLYNOMIAL GROWTH

BY ROMAIN TESSERA

ABSTRACT. — Let G be a compactly generated locally compact group and let U be a compact generating set. We prove that if G has polynomial growth, then $(U^n)_{n\in\mathbb{N}}$ is a Følner sequence and we give a polynomial estimate of the rate of decay of $\frac{\mu(U^{n+1}\smallsetminus U^n)}{\mu(U^n)}$. Our proof uses only two ingredients: the doubling property and a weak geodesic property that we call Property (M). As a matter of fact, the result remains true in a wide class of doubling metric measured spaces including manifolds and graphs. As an application, we obtain a L^p-pointwise ergodic theorem ($1\leq p<\infty$) for the balls averages, which holds for any compactly generated locally compact group G of polynomial growth.

RÉSUMÉ (*Volume de sphères dans les espaces métriques mesurés doublants et dans les groupes à croissance polynomiale*)

Soit G un groupe localement compact, compactement engendré et U une partie compacte génératrice. On prouve que si G est à croissance polynomiale, alors la suite des puissances de U forme une suite de Følner et on montre que le rapport $\frac{\mu(U^{n+1}\smallsetminus U^n)}{\mu(U^n)}$ tend polynomialement vers 0. La démonstration n'utilise que deux ingrédients : le fait qu'un groupe à croissance polynomiale est doublant, et une propriété de faible géodésicité : la propriété (M). Par conséquent ce résultat s'étend à une large classe d'espaces métriques mesurés doublants, comme les graphes et les variétés riemanniennes. Comme application, nous obtenons un théorème ergodique presque sûr et dans L^p ($1\leq p<\infty$) pour les moyennes sur les boules d'un groupe à croissance polynomiale.

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1. Introduction

Let G be a compactly generated, locally compact (cglc) group endowed with a left Haar measure μ. Recall that a sequence $(A_n)_{n\in\mathbb{N}}$ of measurable subsets of a locally compact group G is said to be Følner if for any compact set K,

$$\mu(K\cdot A_n \vartriangle A_n) = o\big(\mu(A_n)\big).$$

Let U be a compact generating set of G (we mean by this that $\bigcup_{n\in\mathbb{N}} U^n = G$), not necessarily symmetric. If $\mu(U^n)$ grows exponentially, it is easy to see that the sequence $(U^n)_{n\in\mathbb{N}}$ *cannot* be Følner. On the other hand, if $\mu(U^n)$ grows subexponentially, then there exists trivially a sequence $(n_i)_{i\in\mathbb{N}}$ of integers such that $(U^{n_i})_{i\in\mathbb{N}}$ is Følner. But it is not clear whether the whole sequence $(U^n)_{n\in\mathbb{N}}$ is Følner. This was first conjectured for amenable groups by Greenleaf in 1969 (see [8, p. 69]), who also proved it with Emerson [7] in the abelian case, correcting a former proof of Kawada [12] (see also Proposition 21). The conjecture is actually not true for all finitely generated amenable groups since there exist amenable groups with exponential growth (for instance, all solvable groups which are not virtually nilpotent). Nevertheless, the conjecture is still open for groups with subexponential growth. In 1983, Pansu [17] proved it for nilpotent finitely generated groups. In [2], Breuillard recently generalized the theorem of Pansu, which now holds for general cglc groups of polynomial growth. In fact, they prove that $\mu(U^n)\sim Cn^d$, for a constant $C = C(U) > 0$, which clearly implies that $(U^n)_{n\in\mathbb{N}}$ is Følner. In this article, we prove the conjecture for all compactly generated groups with polynomial growth. More precisely, we prove the following theorem: there exist $\delta > 0$ and a constant $C < \infty$, such that

$$\mu\left(U^{n+1}\smallsetminus U^n\right) \le Cn^{-\delta}\mu(U^n).$$

Interestingly, our proof works in a much more general setting. Recall that a metric measure space (X, d, μ) satisfies the doubling condition (or "is doubling") if there exists a constant $C \ge 1$ such that

$$\forall r > 0,\ \forall x\in X,\quad \mu\big(B(x, 2r)\big) \le C\mu\big(B(x, r)\big)$$

where $B(x, r) = \{y \in X,\ d(x, y) \le r\}$. Let $S(x, r)$ denote the "1-sphere" of center x and radius r, *i.e.*, $S(x, r) = B(x, r + 1) \smallsetminus B(x, r)$. Actually, we prove a similar result for doubling metric measured spaces satisfying a weak geodesic property we will call Property (M) (see 5.2). In this setting, the result becomes: there exist $\delta > 0$ and a constant $C < \infty$, such that

$$\forall x \in X,\ \forall r > 0,\quad \mu\big(S(x, r)\big) \le Cr^{-\delta}\mu\big(B(x, r)\big).$$

In particular, the conclusion of this theorem holds for metric graphs and Riemannian manifolds satisfying the doubling condition.

In the case of metric measured spaces, our result is somewhat optimal, since in [21, Thm. 4.9], we build a graph X, quasi-isometric to $\mathbb{Z}^2$, such that there exist $0 < a < 1$, an increasing sequence of integers $(n_i)_{i\in\mathbb{N}}$ and $x \in X$ such [(1)] that

$$\forall i \in \mathbb{N}, \quad \big|S(x, n_i)\big| \geq c n_i^{-a} \big|B(x, n_i)\big|.$$

Note that easier counter examples can be obtained with trees with linear growth (see Remark 5). Moreover, we will see that our assumptions on X, that is, Doubling Property and Property (M) (see Definition 1 below) are also optimal in some sense.

An interesting and historical motivation (see for instance [8]) for finding Følner sequences in groups comes from ergodic theory. As a consequence of our result, we obtain a L^p-pointwise ergodic theorem ($1 \leq p < \infty$) for the balls averages, which holds for any cglc group G of polynomial growth (see Theorem 13). We refer to a recent survey of A. Nevo [16] for more details and complete proofs.

2. Main results

2.1. Property (M)

Definition 1. — We say that a metric space (X, d) has *Property (M)* if there exists $C < \infty$ such that the Hausdorff distance between any pair of balls with same center and any radii between r and $r+1$ is less than C. In other words, for all $x \in X$, for all $r > 0$ and for all $y \in B(x, r+1)$, we have $d(y, B(x, r)) \leq C$.

Proposition 2. — *Let (X, d) be a metric space. The following properties are equivalent:*

1) *X has Property (M).*
2) *X has "monotone [(2)] geodesics",* i.e., *there exists $C < \infty$ such that, for all $x, y \in X$, $d(x, y) \geq 1$, there exists a finite chain $x_0 = y$, $x_1, \ldots, x_m = x$ such that for $0 \leq i < m$*

$$d(x_i, x_{i+1}) \leq C \quad \textit{and} \quad d(x_{i+1}, x) \leq d(x_i, x) - 1.$$

3) *There exists a constants $C < \infty$ such that for all $r > 0$, $s \geq 1$ and $y \in B(x, r+s)$*

$$d\big(y, B(x, r)\big) \leq Cs.$$

(1) In our example, $a = \log 2 / \log 3$.
(2) This is why we call this property (M).

Proof

1) $\Rightarrow$ 2). Let $x, y \in X$ be such that $d(x,y) \geq 1$. Let us construct the sequence $y = x_0, x_1, \ldots, x_m = x$ inductively. First, by Property (M) and since $d(x,y) \geq 1$, there exists $x_1 \in B(x, d(x,y)-1)$ such that $d(y,x_1) \leq C$. Now, assume that we have constructed a sequence $y = x_0, x_1, \ldots, x_k$ such that $d(x_i, x_{i+1}) \leq C$ for $0 \leq i < k$, and $d(x_{i+1}, x) \leq d(x_i, x) - 1$. If $d(x_k, x) < 1$, then up to replace C by $C+1$, and x_k by x, the sequence $x_0, \ldots, x_k$ is a monotone geodesic between x and y. Otherwise, there exists $x_{k+1} \in B(x, d(x, x_k) - 1)$ such that $d(x_k, x_{k+1}) \leq C$. Clearly this process has to stop after at most $[d(x,y)]$ steps, so we are done.

2) $\Rightarrow$ 3). Let $x_0 = y, x_1, \ldots, x_m = x$ be a monotone geodesic from y to x. There exists an integer $k \leq s+1$ such that $x_{m-k} \in B(x,r)$. Hence

$$d\big(y, B(x,r)\big) \leq d(y, x_k) \leq Ck \leq C(s+1) \leq 2Cs$$

which proves the implication.

3) $\Rightarrow$ 1). Just take $s = 1$. □

Invariance under Hausdorff equivalence. — Recall (see [10, p. 2]) that two metric spaces X and Y are said Hausdorff equivalent

$$X \sim_{\mathrm{Hau}} Y$$

if there exists a (larger) metric space Z such that X and Y are contained in Z and such that

$$\sup_{x \in X} d(x, Y) < \infty \quad \text{and} \quad \sup_{y \in Y} d(y, X) < \infty.$$

It is easy to see that Property (M) is invariant under Hausdorff equivalence. But on the other hand, Property (M) is unstable under quasi-isometry. To construct a counterexample, one can quasi-isometrically embed $\mathbb{R}_+$ into $\mathbb{R}^2$ such that the image, equipped with the induced metric does not have Property (M): consider a stairway-like curve starting from 0 and containing for every $k \in \mathbb{N}$ a half-circle of radius 2^k centered on 0. So (M) is strictly stronger than the quasi-geodesic property (see [10, p. 7]), which is invariant under quasi-isometry: X is quasi-geodesic if there exist two constants $d > 0$ and $\lambda > 0$ such that for all $(x,y) \in X^2$ there is a finite chain of points $x = x_0, \ldots, x_m = y$, of X such that

$$d(x_{i-1}, x_i) \leq d, \quad i = 1, \ldots, m,$$

and

$$\sum_{i=1}^{m} d(x_{i-1}, x_i) \leq \lambda d(x,y).$$

Note that a monotone geodesic is a quasi-geodesic.