Bulletin
de la SOCIÉTÉ MATHÉMATIQUE DE FRANCE

UNIFORM RESOLVENT ESTIMATES FOR A NON-DISSIPATIVE HELMHOLTZ EQUATION

Julien Royer

Tome 142
Fascicule 4

2014

SOCIÉTÉ MATHÉMATIQUE DE FRANCE
Publié avec le concours du Centre national de la recherche scientifique
pages 591-633

Bull. Soc. math. France
142 (4), 2014, p. 591–633

UNIFORM RESOLVENT ESTIMATES FOR A NON-DISSIPATIVE HELMHOLTZ EQUATION

BY JULIEN ROYER

ABSTRACT. — We study the high frequency limit for a non-selfadjoint Helmholtz equation. This equation models the propagation of the electromagnetic field of a laser in an inhomogeneus material medium with non-constant absorption index. In this paper the absorption index can take negative values and we only use a damping condition on the classical limit of the problem. In this setting we first prove the absence of eigenvalue on the upper half-plane and close to an energy which satisfies this damping assumption. Then we generalize the resolvent estimates of Robert-Tamura and prove the limiting absorption principle. We finally study the semiclassical measures of the solution when the source term concentrates on a bounded submanifold of $\mathbb{R}^n$.

1. Introduction and statement of the main results

The purpose of this paper is to study on $\mathbb{R}^n$, $n \geqslant 1$, the high frequency limit for the Helmholtz equation in a non-dissipative setting. After rescaling, this equation can be written

$$(H_h - E)u_h = f_h, \quad \text{where} \quad H_h = -h^2\Delta + V_1(x) - ihV_2(x). \tag{1.1}$$

We recall that this equation models for instance the propagation of the electromagnetic field of a laser in an inhomogeneus material medium. In this setting

Texte reçu le 29 novembre 2011, révisé le 9 janvier 2013, accepté le 22 mars 2013.

JULIEN ROYER, Institut de mathématiques de Toulouse, 118, route de Narbonne, 31062 Toulouse Cédex 9, France • *E-mail :* `julien.royer@math.univ-toulouse.fr`

2010 Mathematics Subject Classification. — 35J10, 47A10, 47A55, 47B44, 47G30, 81Q20.

Key words and phrases. — Non-selfadjoint operators, resolvent estimates, limiting absorption principle, Helmholtz equation, semiclassical measures.

$V_1(x) - E$ is linked to the refraction index, $V_2(x)$ is the absorption index and f_h is the source term. The parameter $h > 0$ is proportional to the wavelength. In this paper we are interested in the asymptotic behavior of the solution u_h when h goes to 0.

All along this paper, we assume that V_1 and V_2 are bounded and go to 0 at infinity. This implies in particular that the essential spectrum of H_h on the Sobolev space $H^2(\mathbb{R}^n)$ is $\mathbb{R}_+$, as for the free Laplacian. Our purpose is to study the resolvent $(H_h - z)^{-1}$, where $h > 0$ is small and $z \in \mathbb{C}_+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ is close to $E \in \mathbb{R}_+^*$. We prove some estimates for this resolvent uniform in the spectral parameter z, in order to obtain the limiting absorption principle and then existence and uniqueness of an outgoing solution u_h for (1.1). We also control the dependence in h of these estimates, which gives an a priori estimate for the size of u_h when h goes to 0. Note that it is not clear that the resolvent is well-defined. More precisely the operator H_h may have eigenvalues in a strip of size $O(h)$ around the real axis. Therefore we first have to prove that it cannot happen where we study the resolvent.

Let $\delta > \frac{1}{2}$. In the self-adjoint case ($V_2 = 0$) it is known that there exist a neighborhood I of $E > 0$, $h_0 > 0$ and $c \geqslant 0$ such that

$$\forall h \in]0, h_0], \quad \sup_{\substack{\operatorname{Re} z \in I \\ \operatorname{Im} z \neq 0}} \left\| \langle x \rangle^{-\delta} (H_1^h - z)^{-1} \langle x \rangle^{-\delta} \right\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \leqslant \frac{c}{h} \tag{1.2}$$

if and only if the energy E is non-trapping (see (1.5) below). Here we denote by H_1^h the self-adjoint Schrödinger operator $-h^2\Delta + V_1(x)$, by $\mathcal{L}(L^2(\mathbb{R}^n))$ the space of bounded operators on $L^2(\mathbb{R}^n)$, and $\langle x \rangle = \big(1 + |x|\big)^{\frac{1}{2}}$. D. Robert and H. Tamura [23] proved that the non-trapping condition is sufficient and X.P. Wang [31] proved its necessity. In fact, if the non-trapping condition is not satisfied then the norm in (1.2) is at least of size $|\ln h| / h$ (see [4]).

For this result and all along this paper the potential V_1 is assumed to be of long range: it is smooth and there exist constants $\rho > 0$ and $c_\alpha \geqslant 0$ for $\alpha \in \mathbb{N}^n$ such that

$$\forall \alpha \in \mathbb{N}^n, \forall x \in \mathbb{R}^n, \quad |\partial^\alpha V_1(x)| \leqslant c_\alpha \langle x \rangle^{-\rho - |\alpha|}. \tag{1.3}$$

Let $p : (x, \xi) \mapsto \xi^2 + V_1(x)$ be the semiclassical symbol of H_1^h on $\mathbb{R}^{2n} \simeq T^*\mathbb{R}^n$ and ϕ^t the corresponding Hamiltonian flow. For any $w \in \mathbb{R}^{2n}$, $t \mapsto \phi^t(w) = \big(X(t, w), \Xi(t, w)\big)$ is the solution of the system

$$\begin{cases} \partial_t X(t, w) = 2\Xi(t, w), \\ \partial_t \Xi(t, w) = -\nabla V_1(X(t, w)), \\ \phi^0(w) = w. \end{cases} \tag{1.4}$$

We recall that $E > 0$ is said to be non-trapping if

$$\forall w \in p^{-1}(\{E\}), \quad |X(t,w)| \xrightarrow[t\to\pm\infty]{} +\infty. \tag{1.5}$$

For $I \subset \mathbb{R}$, we introduce the following subsets of $p^{-1}(I)$:

$$\begin{aligned}
\Omega_b^{\pm}(I) &= \left\{ w \in p^{-1}(I) \;:\; \sup_{t\geqslant 0} |X(\pm t, w)| < \infty \right\},\\
\Omega_b(I) &= \Omega_b^-(I) \cap \Omega_b^+(I),\\
\Omega_\infty^{\pm}(I) &= \left\{ w \in p^{-1}(I) \;:\; |X(\pm t, w)| \xrightarrow[t\to+\infty]{} +\infty \right\}.
\end{aligned}$$

In [26] we considered the dissipative case $V_2 \geqslant 0$. We proved (1.2) for $\operatorname{Im} z > 0$ under a damping assumption on trapped trajectories:

$$\forall w \in \Omega_b(\{E\}), \exists T \in \mathbb{R}, \quad V_2\big(X(T,w)\big) > 0. \tag{1.6}$$

Notice that this generalizes the usual non-trapping condition: when $V_2 = 0$ then (1.6) is equivalent to (1.5).

To prove this result we developed a dissipative version of Mourre's theory [20], which we applied to the dissipative Schrödinger operator. For this we constructed an escape function as introduced by Ch. Gérard and A. Martinez [14], using the damping assumption to allow trapped trajectories. Note that L. Aloui and M. Khenissi also proved some resolvent estimates for a dissipative Schrödinger operator in [1]. They needed a similar assumption but used a different approach (see below).

We know that assumption (1.6) is both sufficient and necessary in the dissipative setting. Our purpose is now to relax the dissipative condition, allowing negative values for the absorption index V_2. In this case, the damping assumption need reformulating. The condition we are going to use in this paper is the following:

$$\forall w \in \Omega_b(\{E\}), \exists T > 0, \quad \int_0^T V_2(X(t,w))\,dt > 0. \tag{1.7}$$

This condition is in particular satisfied if $V_2 \geqslant 0$ and (1.6) holds. From this point of view, the results we are going to prove here are stronger than those given in the dissipative setting. With Assumption (1.7) and compactness of $\Omega_b(\{E\})$, we can prove that for all $w \in \Omega_b(\{E\})$ we have

$$\liminf_{T\to+\infty} \frac{1}{T} \int_0^T V_2(X(t,w))\,dt > 0,$$

which means that we have a damping condition "on average in time" on bounded trajectories. See Proposition 2.3 for a more precise statement. The relation between asymptotic spectral properties of the non-selfadjoint Helmholtz equation and the average of the absorption index on classical trajectories has already been studied on compact manifolds by J. Sjöstrand in [30].

In this setting we cannot use the dissipative version of Mourre's commutators method. We use the same approach as in [1] instead. The idea is due to G. Lebeau [19] and N. Burq [6]. It is a contradiction argument. We consider a family of functions which denies the result, a semiclassical mesure associated to this family and finally we prove that this measure is both zero and non-zero. This idea was used in [6] for a general self-adjoint and compactly supported perturbation of the Laplacian. In [17], Th. Jecko used the argument to give a new proof of (1.2) with a real-valued potential. The motivation was to give a proof which could be applied to matrix-valued operators. To allow long range potentials, the author introduced a bounded "escape function" which we use here. The method was then used in [8] for a potential with Coulomb singularities and in [18, 13, 10] for a matrix-valued operator.

Let us now state the main results about the resolvent. An important difference with the dissipative case is that we do not know if the resolvent is well-defined, even on the upper half-plane $\mathbb{C}_+$. However, our operator H_h is a relatively compact perturbation of the Laplacian, so according to Weyl's Theorem [21, §XIII.4], its essential spectrum is $\mathbb{R}_+$ and it can only have isolated eigenvalues on $\mathbb{C}_+$. So in the results we state now, we first claim that H_h has no eigenvalue in the considered region and then give an estimate for the resolvent. The first theorem is about spectral parameters whose imaginary parts are bigger than βh for some $\beta > 0$:

THEOREM 1.1. — *Suppose V_2 is smooth with bounded derivatives and $V_2(x) \to 0$ when $|x| \to +\infty$. Let $E > 0$ be an energy which satisfies the damping assumption (1.7) and $\beta > 0$. Then there exist a neighborhood I of E, $h_0 > 0$ and $c \geqslant 0$ such that for $h \in]0, h_0]$ and*

$$z \in \mathbb{C}_{I,h\beta} = \{z \in \mathbb{C} \,:\, \operatorname{Re} z \in I, \operatorname{Im} z \geqslant h\beta\}$$

the operator $(H_h - z) : H^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ has a bounded inverse on $L^2(\mathbb{R}^n)$ and

$$\left\| (H_h - z)^{-1} \right\|_{\mathscr{L}(L^2(\mathbb{R}^n))} \leqslant \frac{c}{h}.$$

Theorem 1.1 is obvious when H_h is self-adjoint or at least dissipative. We can take $c = \beta^{-1}$ in these cases. The statement remains easy in the non-dissipative setting when $\beta > \|V_2\|_\infty$ (see (3.3)), and the point of the theorem is to prove it for any $\beta > 0$. Notice that we can deduce an estimate of size $O(h^{-3})$ for $(H_h - z)^{-1}$ as an operator in $\mathscr{L}(L^2(\mathbb{R}^n), H^2(\mathbb{R}^n))$.