HÖLDER CONTINUITY
OF LYAPUNOV EXPONENT
FOR QUASI-PERIODIC
JACOBI OPERATORS

Kai Tao

HÖLDER CONTINUITY OF LYAPUNOV EXPONENT
FOR QUASI-PERIODIC JACOBI OPERATORS

by Kai Tao

ABSTRACT. — We consider the quasi-periodic Jacobi operator $H_{x,\omega}$ in $l^2(\mathbb{Z})$:
$(H_{x,\omega}\phi)(n) = -b(x+(n+1)\omega)\phi(n+1) - b(x+n\omega)\phi(n-1) + a(x+n\omega)\phi(n) = E\phi(n)$,
$n \in \mathbb{Z}$, where $a(x)$, $b(x)$ are analytic functions on $\mathbb{T}$, b is not identically zero, and ω
obeys some strong Diophantine condition. We consider the corresponding unimodular
cocycle. We prove that if the Lyapunov exponent $L(E)$ of the cocycle is positive
for some $E = E_0$, then there exist $\rho_0 = \rho_0(a,b,\omega,E_0)$, $\beta = \beta(a,b,\omega)$ such that
$|L(E) - L(E')| < |E - E'|^\beta$ for any $E, E' \in (E_0 - \rho_0, E_0 + \rho_0)$. If $L(E) > 0$ for
all E in some compact interval I, then $L(E)$ is Hölder continuous on I with Hölder
exponent $\beta = \beta(a,b,\omega,I)$. In our derivation we follow the refined version of the
Goldstein-Schlag method [3] developed by Bourgain and Jitomirskaya [2].

Kai Tao, College of Sciences, Hohai University, 1 Xikang Road Nanjing Jiangsu 210098
P.R.China • *E-mail :* `ktao@hhu.edu.cn, tao.nju@gmail.com`

1. Introduction

We consider the quasi-periodic Jacobi operator $H_{x,\omega}$ in $l^2(\mathbb{Z})$:

$$(1.1) \quad (H_{x,\omega}\phi)(n)$$
$$= -b(x + (n+1)\omega)\phi(n+1) - b(x+n\omega)\phi(n-1) + a(x+n\omega)\phi(n)$$
$$= E\phi(n), \ n \in \mathbb{Z},$$

where $a(x)$, $b(x)$ are real analytic functions on $\mathbb{T}$, b is not identically zero. Set

$$A(x, E, \omega) = \frac{1}{b(x+\omega)}\begin{pmatrix} a(x) - E & -b(x) \\ b(x+\omega) & 0 \end{pmatrix}.$$

$$M_N(x, E, \omega) = M_{[1,N]}(x, E, \omega)$$
$$= A(x + (N-1)\omega, E, \omega)A(x + (N-2)\omega, E, \omega)\cdots A(x, E, \omega).$$

Define the unimodular matrix

$$\tilde{M}_N(x, E, \omega) = \tilde{M}_{[1,N]}(x, E, \omega) := \frac{M_{[1,N]}(x, E, \omega)}{|\det M_{[1,N]}(x, E, \omega)|^{\frac{1}{2}}}.$$

As

$$\det A(x, E, \omega) = \frac{b(x)}{b(x+\omega)},$$

then

$$(1.2) \qquad \det M_{[1,N]}(x, E, \omega) = \prod_{n=0}^{N-1}\frac{b(x+n\omega)}{b(x+(n+1)\omega)} = \frac{b(x)}{b(x+N\omega)},$$

and

$$(1.3) \ \log\|\tilde{M}_{[1,N]}(x, E, \omega)\| = \log\|M_{[1,N]}(x, E, \omega)\| - \frac{1}{2}\log\left|\frac{b(x)}{b(x+N\omega)}\right|.$$

REMARK 1.1. — (1) Note that

$$\|A(x, E, \omega)\| \le \frac{C(a, b, E)}{|b(x+\omega)|},$$

where the constant $C(a, b, E)$ satisfies

$$C(a, b, E_0) = \sup_{|E| \le E_0} C(a, b, E) < +\infty.$$

Therefore,

$$\frac{1}{N}\log\|M_{[1,N]}(x, E, \omega)\| \le \log C(a, b, E) - \frac{1}{N}\sum_{n=1}^{N}\log|b(x+n\omega)|.$$

In this paper we always assume that $|E| \leq E_0$, where E_0 depends on a, b. For that matter we suppress E from the notation of some of the constants involved.

(2) $\log \|\tilde{M}_{[1,N]}(x, E, \omega)\| \geq 0$, since $\tilde{M}_{[1,N]}(x, E, \omega)$ is unimodular.

(3)

$$
\begin{aligned}
0 \leq \ & \frac{1}{N} \log \|\tilde{M}_{[1,N]}(x, E, \omega)\| \\
= \ & \frac{1}{N} \log \|M_{[1,N]}(x, E, \omega)\| - \frac{1}{2N} \log \left| \frac{b(x)}{b(x + N\omega)} \right| \\
\leq \ & \log C(a, b, E) - \frac{1}{N} \sum_{n=1}^{N} \log |b(x + n\omega)| - \frac{1}{2N} \log \left| \frac{b(x)}{b(x + N\omega)} \right|.
\end{aligned}
$$

(4) It is a well-known fact that if b is an analytic function not identically zero, then $(\log |b|)^2$ is integrable. Set

$$
D = \int_{\mathbb{T}} \log |b(\theta)| d\theta.
$$

Therefore,

$$
\begin{aligned}
\int_{\mathbb{T}} \left| \frac{1}{N} \log \|\tilde{M}_{[1,N]}(x, E, \omega)\| \right| dx = \ & \int_{\mathbb{T}} \frac{1}{N} \log \|\tilde{M}_{[1,N]}(x, E, \omega)\| dx \\
\leq \ & C'(a, b) - D := C''(a, b).
\end{aligned}
$$

Similarly,

$$
\int_{\mathbb{T}} \left(\frac{1}{N} \log \|\tilde{M}_N(x, E, \omega)\| \right)^2 dx \leq \tilde{C}(a, b).
$$

(5) Combining (4) with (1.3), we conclude that $\frac{1}{N} \log \|M_{[1,N]}(x, E, \omega)\|$ is integrable, and

$$
\frac{1}{N} \int_{\mathbb{T}} \log \|M_N(x, E, \omega)\| dx = \frac{1}{N} \int_{\mathbb{T}} \log \|\tilde{M}_N(x, E, \omega)\| dx.
$$

Set

$$
L_N(E, \omega) = \frac{1}{N} \int_{\mathbb{T}} \log \|M_N(x, E, \omega)\| dx = \frac{1}{N} \int_{\mathbb{T}} \log \|\tilde{M}_N(x, E, \omega)\| dx.
$$

Note that $L_N(E, \omega) > 0$. Set

$$
B(x, E, \omega) := \begin{pmatrix} a(x) - E & -b(x) \\ b(x + \omega) & 0 \end{pmatrix},
$$

and

$$
\begin{aligned}
T_N(x, E, \omega) = \ & T_{[1,N]}(x, E, \omega) \\
:= \ & B(x + (N-1)\omega, E, \omega) B(x + (N-2)\omega, E, \omega) \cdots B(x, E, \omega).
\end{aligned}
$$

Then

$$M_{[1,N]}(x, E, \omega) = T_{[1,N]}(x, E, \omega) \prod_{n=N-1}^{0} \frac{1}{b(x + (n+1)\omega)},$$

(1.4)
$$\tilde{M}_{[1,N]}(x, E, \omega) = \frac{|b(x + N\omega)|^{\frac{1}{2}}}{|b(x)|^{\frac{1}{2}}} M_{[1,N]}(x, E, \omega)$$

$$= \frac{|b(x + N\omega)|^{\frac{1}{2}}}{|b(x)|^{\frac{1}{2}}} \prod_{n=0}^{N-1} \frac{1}{b(x + (n+1)\omega)} T_{[1,N]}(x, E, \omega),$$

(1.5) $\|\tilde{M}_{[1,N]}(x, E, \omega)\| = \displaystyle\prod_{n=0}^{N-1} \frac{1}{|b(x + n\omega)b(x + (n+1)\omega)|^{\frac{1}{2}}} \|T_{[1,N]}(x, E, \omega)\|,$

and

$$\log \|\tilde{M}_{[1,N]}(x, E, \omega)\| = \log \|T_{[1,N]}(x, E, \omega)\| - \frac{1}{2} \sum_{n=0}^{N-1} \log |b(x + n\omega)b(x + (n+1)\omega)|.$$

Note also for future reference that

(1.6) $\qquad | \det T_{[1,N]}(x, E, \omega)| = \displaystyle\prod_{n=0}^{N-1} |b(x + n\omega)||b(x + (n+1)\omega)|.$

Combining (1) with Remark 1.1, we conclude that $\frac{1}{N} \log \|T_{[1,N]}(x, E, \omega)\|$ is integrable,

(1.7) $\qquad J_N(E, \omega) := \dfrac{1}{N} \displaystyle\int_{\mathbb{T}} \log \|T_{[1,N]}(x, \omega)\| dx = L_N(E, \omega) + D.$

Due to the subadditive property, the limits

(1.8)
$$L(E, \omega) = \lim_{N \to \infty} \int_{\mathbb{T}} \frac{1}{N} \log \|M_N(x, E, \omega)\| dx = \lim_{N \to \infty} \int_{\mathbb{T}} \frac{1}{N} \log \|\tilde{M}_N(x, E, \omega)\| dx$$
$$= \lim_{N \to \infty} L_N(E, \omega),$$

(1.9)
$$J(E, \omega) = \lim_{N \to \infty} \int_{\mathbb{T}} \frac{1}{N} \log \|T_N(x, E, \omega)\| dx = \lim_{N \to \infty} J_N(E, \omega) = L(E, \omega) + D$$

exist. Moreover, $L(E, \omega) \geq 0$. Fix some $\alpha > 1$. Throughout this paper we assume that $\omega \in (0, 1)$ satisfies the Diophantine condition

(1.10) $\qquad\qquad\qquad \|n\omega\| \geq \dfrac{C_\omega}{n(\log n)^\alpha} \quad \text{for all } n.$

It is well known that for a fixed $\alpha > 1$ almost every ω satisfies (1.10).

The main theorem in this paper is