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FAMILY OF INTERSECTING TOTALLY REAL MANIFOLDS OF $(\mathbb{C}^n, 0)$ AND GERMS OF HOLOMORPHIC DIFFEOMORPHISMS

by Laurent Stolovitch

Abstract. — We prove the existence (and give a characterization) of a germ of complex analytic set left invariant by an abelian group of germs of holomorphic diffeomorphisms at a common fixed point.We also give condition that ensure that such a group can be linearized holomorphically near the fixed point. It rests on a "small divisors condition" of the family of linear parts.

The second part of this article is devoted to the study families of totally real intersecting n-submanifolds of $(\mathbb{C}^n, 0)$. We give some conditions which allow to straighten holomorphically the family. If it is not possible to do this formally, we construct a germ of complex analytic set at the origin which intersection with the family can be holomorphically straightened. The second part is an application of the first.

1. Introduction

One of the aim of the article is to study the geometry of some germs of real analytic submanifolds of $(\mathbb{C}^n, 0)$. We shall consider, in this article, only families of totally real submanifolds of $(\mathbb{C}^n, 0)$ intersecting at the origin. We are primarily interested in the holomorphic classification of such objects, that is the orbits of the action of the group of germs of holomorphic diffeomorphisms fixing the origin.

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In this article, we shall mainly focus on the existence of complex analytic subsets intersecting such germs of real analytic manifolds. We shall also be interested in the problem of straightening holomorphically the family. We mean that we shall give sufficient condition which will ensure that, in a good holomorphic coordinates system, each (germ of) submanifold of the family is an n-plane. In the case there are formal obstructions to straighten the family, we show the existence of a germ complex analytic variety which intersects the family along a set that can be straightened. This part of this work takes its roots in and generalizes a work of Sidney Webster [21] from which it is very inspired. This part of the work start after having listen to Sidney Webster at the Partial Differential Equations and Several Complex Variables conference held in Wuhan University in June 2004.

The starting point of the first problem appeared already in the work of E. Kasner [8] and was studied, from the formal view point, by G.A. Pfeiffer [12]. They were interested in pairs of real analytic curves in $(\mathbb{C}, 0)$ passing through the origin. We shall not consider the case were some of the submanifolds are tangent to some others. We refer the reader to the works of I. Nakai [11], J.-M. Trépreau [20] and P. Ahern and X. Gong [1] in this direction.

The core of this problem rests on geometric properties of an associated dynamical systems. To be more specific, we shall deal, in the first part of this article, with germs of holomorphic diffeomorphisms of $(\mathbb{C}^n, 0)$ in a neighborhood of the origin (a common fixed point). We shall consider those whose linear part at the origin is different from the identity. The main result is the existence of germ of analytic subset of $(\mathbb{C}^n, 0)$ invariant by an abelian group of such diffeomorphisms under some diophantine condition. This diophantine condition is a Brjuno like condition over small divisors of the full family of linear parts (such condition was already devised by the author for commuting vector fields in [17]). The main result is obtained when trying not to linearize (this not possible in general) but rather to linearize along a well-chosen ideal in a neighborhood of the origin. In fact, this is almost always possible when considering namely the *resonant ideal*, generated by the polynomial first integrals of the linear part. The zero locus of this ideal provides, in good holomorphic coordinates, the invariant analytic set. If the family is formally linearizable and the family of linear parts satisfies the small divisor condition then, we shall also prove the family is holomorphically linearizable (in that case the ideal is chosen to be zero). This first kind (resp. second) of result was obtained by the author for a single (resp. family of commuting) germ of holomorphic vector field at singular point [16] (resp. [17]). This article corresponds to the two first parts of the preprint [19]. This work is also used in a recent work in collaboration with Xianghong Gong [5].

2. Abelian group of diffeomorphisms of $(\mathbb{C}^n,0)$ and their invariant sets

The aim of this section is to prove the existence of complex analytic invariant subset for a commuting family of germs of holomorphic diffeomorphisms in a neighborhood of a common fixed point. This is very inspired by a previous article of the author concerning holomorphic vector fields. Although the objects are not the same, some of the computations are identical and we shall refer to them when possible.

Let $D_1 := \mathrm{diag}(\mu_{1,1},\dots,\mu_{1,n}),\dots,D_l := \mathrm{diag}(\mu_{l,1},\dots,\mu_{l,n})$ be diagonal invertible matrices. Let us consider a family $F := \{F_i\}_{i=1,\dots,l}$ of *commuting germs of holomorphic diffeomorphisms* of $(\mathbb{C}^n,0)$ whose linear part, at the origin, is $D := \{D_i x\}_{i=1,\dots,l}$:

$$F_i(x) = D_i x + f_i(x), \quad \text{with} \quad f_i(0)=0,\ Df_i(0)=0,\ f_i\in\mathcal{O}_n.$$

Let $\mathcal{I}$ be an ideal of $\mathcal{O}_n$ *generated by monomials* of $\mathbb{C}^n$. Let $V(\mathcal{I})$ be the germ at the origin, of the analytic subset of $(\mathbb{C}^n,0)$ defined by $\mathcal{I}$. It is left invariant by the family D. Let us set $\hat{\mathcal{I}} := \widehat{\mathcal{O}}_n\otimes\mathcal{I}$. Here we denote $\mathcal{O}_n$ (resp. $\widehat{\mathcal{O}}_n$) the ring of germ of holomorphic function at the origin (resp. ring of formal power series) of $\mathbb{C}^n$. For $Q=(q_1,\dots,q_n)\in\mathbb{N}^n$ and $x=(x_1,\dots,x_n)\in\mathbb{C}^n$, we shall write

$$|Q| := q_1+\dots+q_n,\quad x^Q := x_1^{q_1}\cdots x_n^{q_n}.$$

We shall shall denote $\mathbb{N}_2^n$, the set of $Q\in\mathbb{N}^n$ such that $|Q|\geq 2$.

Let $\{\omega_k(D,\mathcal{I})\}_{k\geq 1}$ be the sequence of positive numbers defined by

$$\omega_k(D,\mathcal{I}) = \inf\left\{\max_{1\leq i\leq l}|\mu_i^Q-\mu_{i,j}|\neq 0 \mid 2\leq|Q|\leq 2^k, 1\leq j\leq n, Q\in\mathbb{N}^n, x^Q\notin\mathcal{I}\right\}$$

where $\mu_i^Q := \mu_{i,1}^{q_1}\cdots\mu_{i,n}^{q_n}$. Let $\{\omega_k(D)\}_{k\geq 1}$ be the sequence of positive numbers defined by

$$\omega_k(D) = \inf\left\{\max_{1\leq i\leq l}|\mu_i^Q-\mu_{i,j}|\neq 0 \mid 2\leq|Q|\leq 2^k, 1\leq j\leq n, Q\in\mathbb{N}^n\right\}.$$

Definition 2.1. — 1. *We say that the ideal $\mathcal{I}$ is properly embedded if it has a set of monomial generators that does not involve all variables. In that case, the set $\mathcal{S}$ of variables not involved in any generator is not empty.*

2. *We say that the family D is* diophantine *(resp. on $\mathcal{I}$) if*

$$(\omega(D)):\quad -\sum_{k\geq 1}\frac{\ln\omega_k(D)}{2^k}<+\infty\quad (\textit{resp. } (\omega(D,\mathcal{I})):\quad -\sum_{k\geq 1}\frac{\ln\omega_k(D,\mathcal{I})}{2^k}<+\infty).$$

3. *A linear anti-holomorphic involution of $\mathbb{C}^n$ is a map $\rho(z)=P\bar{z}$ where the matrix P satisfies $P\bar{P}=\mathrm{Id}$; $\bar{z}$ denotes the complex conjugate of z.*

4. *We denote by $\widehat{CI}$ the vector subspace of formal power series with no monomial in $\mathcal{I}$.*