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RINGS OF MICRODIFFERENTIAL OPERATORS FOR ARITHMETIC $\mathscr{D}$-MODULES — CONSTRUCTION AND AN APPLICATION TO THE CHARACTERISTIC VARIETIES FOR CURVES

BY TOMOYUKI ABE

ABSTRACT. — One aim of this paper is to develop a theory of microdifferential operators for arithmetic $\mathscr{D}$-modules. We first define the rings of microdifferential operators of arbitrary levels on arbitrary smooth formal schemes. A difficulty lies in the fact that there is no homomorphism between rings of microdifferential operators of different levels. To remedy this, we define the intermediate differential operators, and using these, we define the ring of microdifferential operators for $\mathscr{D}^{\dagger}$. We conjecture that the characteristic variety of a $\mathscr{D}^{\dagger}$-module is computed as the support of the microlocalization of a $\mathscr{D}^{\dagger}$-module, and prove it in the curve case.

Introduction

This paper is aimed to construct a theory of rings of microdifferential operators for arithmetic $\mathscr{D}$-modules. Let X be a smooth variety over $\mathbb{C}$. Then the sheaf of rings of microdifferential operators denoted by $\mathscr{E}_X$ is defined on the cotangent bundle T^*X of X. This ring is one of basic tools to study $\mathscr{D}$-modules microlocally, and it is used in various contexts. One of the most important and fundamental properties is the equality

$$\mathrm{Char}(\mathscr{M}) = \mathrm{Supp}(\mathscr{E}_X \otimes_{\pi^{-1}\mathscr{D}_X} \mathscr{M}) \tag{い}$$

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for a coherent $\mathscr{D}_X$-module $\mathscr{M}$, where $\pi : T^*X \to X$ is the projection. One goal of this study is to find an analogous equality in the theory of arithmetic $\mathscr{D}$-modules.

We should point out two attempts to construct rings of microdifferential operators. The first attempt was made by R. G. López in [13]. In there, he constructed the ring of microdifferential operators of *finite order on curves*. However, the relation between his construction and the theory of arithmetic $\mathscr{D}$-modules was not clear as he pointed out in the last remark of [13]. The second construction was carried out by A. Marmora in [23]. Our work can be seen as a generalization of this work, and we explain the relation with our construction in the following.

Now, let R be a complete discrete valuation ring of mixed characteristic $(0,p)$. Let $\mathscr{X}$ be a smooth formal scheme over $\mathrm{Spf}(R)$, and we denote the special fiber of $\mathscr{X}$ by X. For an integer $m \geq 0$, P. Berthelot defined the ring of differential operators of level m denoted by $\widehat{\mathscr{D}}^{(m)}_{\mathscr{X},\mathbb{Q}}$. He also defined the characteristic varieties for coherent $\widehat{\mathscr{D}}^{(m)}_{\mathscr{X},\mathbb{Q}}$-modules in almost the same way we define the characteristic varieties for *analytic* (or *algebraic*) $\mathscr{D}$-modules. It is natural to hope that there exists a theory of microdifferential operators, and that we can define the ring of microdifferential operators $\widehat{\mathscr{E}}^{(m)}_{\mathscr{X},\mathbb{Q}}$ of level m associated with $\widehat{\mathscr{D}}^{(m)}_{\mathscr{X},\mathbb{Q}}$ satisfying an analog of (い). When $\mathscr{X}$ is a curve (and $m = 0$), this was done by Marmora in his study of Fourier transform. He fixed a system of local coordinates, constructed the ring of microdifferential operators using explicit descriptions as in [8, Chapter VIII], and proved that the construction does not depend on the choice of local coordinates. In this paper, we use a general technique of G. Laumon of formal microlocalization of certain filtered rings (cf. [22]) to define the ring of naive microdifferential operators of level m denoted by $\widehat{\mathscr{E}}^{(m)}_{\mathscr{X},\mathbb{Q}}$ (cf. 2.9). One advantage of this construction is that we do not need to choose coordinates. It follows also formally using the result of Laumon that for a coherent $\widehat{\mathscr{D}}^{(m)}_{\mathscr{X},\mathbb{Q}}$-module $\mathscr{M}$, we get an analogous equality of (い)

$$(ろ) \qquad \mathrm{Char}^{(m)}(\mathscr{M}) = \mathrm{Supp}(\widehat{\mathscr{E}}^{(m)}_{\mathscr{X},\mathbb{Q}} \otimes_{\pi^{-1}\widehat{\mathscr{D}}^{(m)}_{\mathscr{X},\mathbb{Q}}} \pi^{-1}\mathscr{M})$$

in $T^{(m)*}X := \mathrm{Spec}(\mathrm{gr}(\mathscr{D}^{(m)}_X))$, where $\pi\colon T^{(m)*}X \to X$ is the projection, and $\mathrm{Char}^{(m)}$ denotes the characteristic variety (cf. 2.14).

Before explaining the construction of sheaves of microdifferential operators associated with $\mathscr{D}^{\dagger}_{\mathscr{X},\mathbb{Q}}$, let us review the theory of Berthelot, and see why we need to consider $\mathscr{D}^{\dagger}_{\mathscr{X},\mathbb{Q}}$-modules. Berthelot proved that many fundamental theorems in the theory of analytic $\mathscr{D}$-modules hold also for $\widehat{\mathscr{D}}^{(m)}_{\mathscr{X},\mathbb{Q}}$-modules. For example, he defined pull-backs and push-forwards, and proved that push-forwards of coherent modules by proper morphisms remain coherent (cf. [7]). However, the analogue of Kashiwara's theorem, which states an equivalence between the category of coherent $\widehat{\mathscr{D}}^{(m)}_{\mathscr{X},\mathbb{Q}}$-modules which are supported on a smooth closed formal subscheme $\mathscr{Z}$ of $\mathscr{X}$ and the category of coherent $\widehat{\mathscr{D}}^{(m)}_{\mathscr{Z},\mathbb{Q}}$-modules, does not hold. This failure makes it difficult to define a suitable subcategory of holonomic modules in the category of $\widehat{\mathscr{D}}^{(m)}_{\mathscr{X},\mathbb{Q}}$-modules. To remedy this, Berthelot took inductive limit on the levels to define the ring $\mathscr{D}^{\dagger}_{\mathscr{X},\mathbb{Q}}$, and proved an analogue of Kashiwara's theorem for coherent $\mathscr{D}^{\dagger}_{\mathscr{X},\mathbb{Q}}$-modules (cf. [7, 5.3.3]). As in the analytic $\mathscr{D}$-module theory, we need to consider holonomic modules to deal with push-forwards along open immersions, and we need to define characteristic varieties to define holonomic modules. When a coherent $\mathscr{D}^{\dagger}_{\mathscr{X},\mathbb{Q}}$-module possesses a *Frobenius structure* (*i.e.,* an isomorphism $\mathscr{M} \xrightarrow{\sim} F^*\mathscr{M}$), Berthelot defined the characteristic variety. He reduced the definition to a finite level situation using a marvelous theorem of Frobenius descent, and proved Bernstein's inequality by using the analogue of Kashiwara's theorem. However, in the absence of Frobenius, the situation is mysterious.

In this paper, we propose a new formalism which allows us at least conjecturally to interpret this characteristic varieties by means of microlocalizations, and use them to define the characteristic varieties for general coherent $\mathscr{D}^{\dagger}_{\mathscr{X},\mathbb{Q}}$-modules which may not carry Frobenius structures. We also prove the conjecture in the case of curves (cf. Theorem 7.2). Let us describe a more precise statement and difficulties to carry this out.

One of the difficulties in defining microdifferential operators associated with $\mathscr{D}^{\dagger}$ is that there are *no* transition homomorphism (cf. 4.1)

$$\widehat{\mathscr{E}}^{(m)}_{\mathscr{X},\mathbb{Q}} \to \widehat{\mathscr{E}}^{(m+1)}_{\mathscr{X},\mathbb{Q}}$$

compatible with $\widehat{\mathscr{D}}^{(m)}_{\mathscr{X},\mathbb{Q}} \to \widehat{\mathscr{D}}^{(m+1)}_{\mathscr{X},\mathbb{Q}}$. This makes it hard to define the ring of microdifferential operators corresponding to $\mathscr{D}^{\dagger}_{\mathscr{X},\mathbb{Q}}$ in a naive way. Let $\pi : T^*\mathscr{X} \to \mathscr{X}$ be the projection. To remedy this, we define a $\pi^{-1}\widehat{\mathscr{D}}^{(m)}_{\mathscr{X},\mathbb{Q}}$-algebra $\widehat{\mathscr{E}}^{(m,m')}_{\mathscr{X},\mathbb{Q}}$ for any integer $m' \geq m$ called the "intermediate ring of microdifferential operators of level (m,m')" so that there exist homomorphisms of $\pi^{-1}\widehat{\mathscr{D}}^{(m)}_{\mathscr{X},\mathbb{Q}}$-algebras

$$\widehat{\mathscr{E}}^{(m,m'+1)}_{\mathscr{X},\mathbb{Q}} \to \widehat{\mathscr{E}}^{(m,m')}_{\mathscr{X},\mathbb{Q}}, \qquad \widehat{\mathscr{E}}^{(m,m')}_{\mathscr{X},\mathbb{Q}} \to \widehat{\mathscr{E}}^{(m+1,m')}_{\mathscr{X},\mathbb{Q}},$$

and $\widehat{\mathscr{E}}^{(m,m)}_{\mathscr{X},\mathbb{Q}} = \widehat{\mathscr{E}}^{(m)}_{\mathscr{X},\mathbb{Q}}$. We define

$$\mathscr{E}^{(m,\dagger)}_{\mathscr{X},\mathbb{Q}} := \varprojlim_{m'} \widehat{\mathscr{E}}^{(m,m')}_{\mathscr{X},\mathbb{Q}}.$$