$L^p$ ESTIMATES FOR MULTI-LINEAR AND MULTI-PARAMETER PSEUDO-DIFFERENTIAL OPERATORS

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1. Introduction

1.1. Background. — For $n \geq 1$ and $d \geq 1$, let $m$ be a bounded function in $\mathbb{R}^{nd}$, smooth away from the origin and satisfying Hörmander-Mikhlin conditions\(^{(1)}\)

\[
|\partial^\alpha m(\xi)| \lesssim \frac{1}{|\xi|^{\alpha}}
\]
for sufficiently many multi-indices $\alpha$. Denote by $T_m$ the $n$-linear operator defined by

\[
T_m(f_1, \ldots, f_n)(x) := \int_{\mathbb{R}^{nd}} m(\xi) \hat{f}_1(\xi) \cdots \hat{f}_n(\xi) e^{2\pi i x \cdot (\xi_1 + \cdots + \xi_n)} d\xi,
\]

where $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^{nd}$ and $f_1, \ldots, f_n$ are Schwartz functions on $\mathbb{R}^d$.

From the classical Coifman-Meyer theorem (see [6, 7, 19, 11]), we know that the operator $T_m$ extends to a bounded $n$-linear operator from $L^{p_1}(\mathbb{R}^d) \times \cdots \times L^{p_n}(\mathbb{R}^d)$ into $L^p(\mathbb{R}^d)$, provided that $1 < p_1, \ldots, p_n \leq \infty$ and $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_n} > 0$. When $n = 2$, as a consequence of bilinear $T1$ theorem (see [6, 11]), there is also a pseudo-differential variant of the classical Coifman-Meyer theorem for symbol $a \in BS^0_{1, 0}(\mathbb{R}^{3d})$, that is, $a$ satisfies the differential inequalities

\[
|\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma a(x, \xi, \eta)| \lesssim_{d, \alpha, \beta, \gamma} (1 + |\xi| + |\eta|)^{-|\alpha| - |\beta|}
\]

for sufficiently many multi-indices $\alpha, \beta, \gamma$. Namely, let $T_a$ be the corresponding bilinear pseudo-differential operators defined by replacing $m$ with $a$ in (1.2), then $T_a$ is bounded from $L^p(\mathbb{R}^d) \times L^q(\mathbb{R}^d)$ into $L^{r}(\mathbb{R}^d)$, provided that $1 < p, q \leq \infty$ and $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} > 0$ (see [2], and see [3, 26, 23] for $d = 1$ case).

For large amounts of literature involving estimates for multi-linear Calderón-Zygmund operators and multi-linear pseudo-differential operators, refer to e.g., [1, 6, 19, 9, 11, 12, 15, 23, 24].

However, when we come into the situation that a differential operator (with different behaviors on different spatial variables $x_i$, $i = 1, \ldots, d$) acts on a product of several functions (for instance, the bilinear form $\mathcal{D}_1^\alpha D_2^\beta(fg)$, where $\mathcal{D}_1^\alpha f(\xi_1, \xi_2) := |\xi_1|^\alpha f(\xi_1, \xi_2)$ and $\mathcal{D}_2^\beta f(\xi_1, \xi_2) := |\xi_2|^\beta f(\xi_1, \xi_2)$ for $\alpha, \beta > 0$), we realize that the necessity to investigate bilinear and bi-parameter operators

\[\text{(1.1)}\]

throughout this paper, $A \lesssim B$ means that there exists a universal constant $C > 0$ such that $A \leq CB$. If necessary, we use explicitly $A \lesssim_1, \ldots, \lesssim_n B$ to indicate that there exists a positive constant $C_1, \ldots, C_n$, depending only on the quantities appearing in the subscript continuously such that $A \leq C_1, \ldots, C_n B$.

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$T_m^{(2)}$ defined by

$$T_m^{(2)}(f, g)(x) := \int_{\mathbb{R}^4} m(\xi, \eta)f(\xi)\hat{g}(\eta)e^{2\pi i x \cdot (\xi + \eta)}d\xi d\eta,$$

where the symbol $m$ is smooth away from the planes $(\xi_1, \eta_1) = (0, 0)$ and $(\xi_2, \eta_2) = (0, 0)$ in $\mathbb{R}^2 \times \mathbb{R}^2$ and satisfying the less restrictive Marcinkiewicz conditions

$$|\partial_{\xi_1}^\alpha \partial_{\eta_1}^\beta \partial_{\xi_2}^\gamma \partial_{\eta_2}^\delta m(\xi, \eta)| \lesssim \frac{1}{|\xi_1^\alpha \eta_1^\beta|} \cdot \frac{1}{|\xi_2^\gamma \eta_2^\delta|}$$

for sufficiently many multi-indices $\alpha = (\alpha_1, \alpha_2)$, $\beta = (\beta_1, \beta_2)$. It becomes more complicated and difficult to establish the $L^p$ estimates for $T_m^{(2)}$ than in the one-parameter multilinear situations or $L^p$ estimates for linear multi-parameter singular integrals (see e.g., [8] and [14]). In [21], by using the duality lemma of $L^{p, \infty}$ presented in [24], the $L^{1, \infty}$ sizes and energies technique developed in [25] and multi-linear interpolation (see e.g., [10, 25]), Muscalu, Pipher, Tao and Thiele proved the following $L^p$ estimates for $T_m^{(2)}$ (see also [23], and for subsequent endpoint estimates see [16]).

**Theorem 1.1** ([21]). — The bilinear operator $T_m^{(2)}$ defined by (1.4) maps $L^p(\mathbb{R}^2) \times L^q(\mathbb{R}^2) \rightarrow L^r(\mathbb{R}^2)$ boundedly, as long as $1 < p, q \leq \infty$ and $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} > 0$.

In general, any collection of $n$ generic vectors $\xi_1 = (\xi_1^i)_{i=1}^d, \ldots, \xi_n = (\xi_n^i)_{i=1}^d$ in $\mathbb{R}^d$ generates naturally the following collection of $d$ vectors in $\mathbb{R}^n$:

$$\hat{\xi}_1 = (\xi_1^1)^n, \ldots, \hat{\xi}_d = (\xi_d^i)^n$$

Let $m = m(\xi) = m(\hat{\xi})$ be a bounded symbol in $L^{\infty}(\mathbb{R}^dn)$ that is smooth away from the subspaces $\{\xi_1 = 0\} \cup \cdots \cup \{\xi_d = 0\}$ and satisfying

$$|\partial_{\xi_1}^\alpha \cdots \partial_{\xi_d}^\alpha m(\hat{\xi})| \lesssim \prod_{i=1}^d |\hat{\xi}_i|^{-|\alpha_i|}$$

for sufficiently many multi-indices $\alpha_1, \ldots, \alpha_d$. Denote by $T_m^{(d)}$ the $n$-linear multiplier operator defined by

$$T_m^{(d)}(f_1, \ldots, f_n)(x) := \int_{\mathbb{R}^{dn}} m(\xi)\hat{f}_1(\xi_1)\cdots\hat{f}_n(\xi_n)e^{2\pi i x \cdot (\xi_1 + \cdots + \xi_n)}d\xi.$$ 

In [22], Muscalu, Pipher, Tao and Thiele generalized Theorem 1.1 to the $n$-linear and $d$-parameter setting for any $n \geq 1, d \geq 2$, their result is stated in the following theorem (see also [23]).