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# RESOLVENT EXPANSIONS AND CONTINUITY OF THE SCATTERING MATRIX AT EMBEDDED THRESHOLDS: THE CASE OF QUANTUM WAVEGUIDES

by S. Richard & R. Tiedra de Aldecoa

Abstract. — We present an inversion formula which can be used to obtain resolvent expansions near embedded thresholds. As an application, we prove for a class of quantum waveguides the absence of accumulation of eigenvalues and the continuity of the scattering matrix at all thresholds.

Résumé (*Expansions de résolventes et continuité de la matrice de diffusion aux seuils immergés: le cas des guides d'onde quantiques*)

Nous présentons une formule d'inversion qui peut être utilisée pour obtenir des expansions de résolventes à proximité de seuils immergés. Comme application, nous démontrons pour une classe de guides d'onde quantiques l'absence d'accumulation de valeurs propres et la continuité de la matrice de diffusion en chaque seuil.

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## 1. Introduction

During the recent years, there has been an increasing interest in resolvent expansions near thresholds and their various applications. These developments were partially initiated by the paper of A. Jensen and G. Nenciu [10] in which a general framework for asymptotic expansions is presented and then applied to potential scattering in dimension 1 and 2. The key point of that paper is an inversion formula which provides an efficient iterative method for inverting a family of operators $A(z)$ as $z \to 0$ even if $\ker\big(A(0)\big) \neq \{0\}$. Corrections or improvements of this inversion formula can be found in [4, Lemma 4], [8, Prop. 3.2] and [11, Prop. 1]. However, in all these papers either it is assumed that $A(0)$ is self-adjoint, or the construction relies on a Riesz projection which is not always convenient to deal with. These features are harmless in these works, since the threshold considered always lies at the endpoints of the spectrum of the underlying operator. However, once dealing with embedded thresholds, these features turn out to be critical (see the comment at the end of Section 2.2).

Our aim in the present paper is thus twofold. On the one hand, we revisit the mentioned inversion formula, and on the other hand we show how its revised version can be used for proving the continuity of a scattering matrix at embedded thresholds. The abstract part of our results is presented in Section 2, and consists first in a reformulation of the inversion formula which does not require that $A(0)$ is self-adjoint or that the projection is a Riesz projection (see Proposition 2.1). We then discuss two natural choices for the projection: either the Riesz projection defined in terms of the resolvent of $A(0)$ if 0 is an isolated point in the spectrum of $A(0)$, or the orthogonal projection on $\ker\big(A(0)\big)$ if $A(0)$ has a non-negative imaginary part. If both conditions hold, we also discuss the relations between these two projections, and provide sufficient conditions for their equality. This situation often takes place in applications even without the assumption that $A(0)$ is self-adjoint (see Corollary 2.8).

In the second part of the paper (Section 3), we present an application of our abstract results to scattering theory for quantum waveguides. Quantum waveguides provide a particularly good model of study since their Hamiltonians possess an infinite number of embedded thresholds (with a change of multiplicity at each threshold) but give rise to a simple scattering theory taking place in a one-Hilbert space setting. We refer to [1] for basic results and earlier references on the spectral and scattering theory for quantum waveguides.

For a straight quantum waveguide with a compactly supported potential $V$, we derive an asymptotic expansion of the resolvent in a neighborhood of each embedded threshold. More precisely, if the potential is written as $V = vuv$ with $v$ non-negative and $u$ unitary and self-adjoint, and if $H_0$ is the Dirichlet Laplacian for the waveguide, then we give an expansion of the operator

$\big(u+v(H_0-z)^{-1}v\big)^{-1}$ as $z$ converges to any threshold $z_0$ (see Proposition 3.2). Note that the operator $v(H_0-z_0)^{-1}v$ (once properly defined) has a non-trivial imaginary part. This fact automatically prevents the use of any approach assuming the self-adjointness of $A(0)$, as mentioned above.

We then deduce two consequences of this asymptotic expansion. First, we prove in Corollary 3.3 that the possible point spectrum of the operator $H := H_0 + V$ does not accumulate at thresholds. Since the thresholds are the only possible accumulation points for such a model, we thus rule out this possibility. Second, we characterize for all scattering channels corresponding to the transverse modes of the waveguide the behavior of the scattering matrix for the pair $\{H_0, H\}$ at embedded thresholds. More precisely, we show that the scattering matrix is continuous at the thresholds if the channels we consider are already open, and that the scattering matrix has a limit from the right at the thresholds if a channel precisely opens at these thresholds (see Proposition 3.8 for a precise formulation of this result). Up to our knowledge, these types of results are completely new since the analysis of the behavior of a scattering matrix at embedded thresholds has apparently never been performed. We also show the continuity of the scattering matrix at embedded eigenvalues which are not located at thresholds. But in this case, similar results were already known for other models, see for example [7, Prop. 10] or [14, Prop. 6.7.11] (see also [5] where propagation estimates at embedded thresholds are obtained for a Schrödinger operator with time periodic potential).

As a final comment, we stress that we fully describe all possible behaviors at thresholds since we do not assume any condition on the absence of bound states or resonances at thresholds. Based on the expressions obtained in this paper, a Levinson's type theorem for quantum waveguides could certainly be derived, and deserves further investigations.

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## 2. Inversion formula

In this section, we adapt the inversion formula [11, Prop. 1] to the case of an arbitrary projection, and then discuss two possible choices for this projection. The symbol $\mathscr{H}$ stands for an arbitrary Hilbert space with norm $\|\cdot\|$ and scalar product $\langle\cdot,\cdot\rangle$, and $\mathscr{B}(\mathscr{H})$ denotes the algebra of bounded operators on $\mathscr{H}$ with norm also denoted by $\|\cdot\|$.