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TWO DIMENSIONAL WATER WAVES IN HOLOMORPHIC COORDINATES II: GLOBAL SOLUTIONS

by Mihaela Ifrim & Daniel Tataru

Abstract. — This article is concerned with the infinite depth water wave equation in two space dimensions. We consider this problem expressed in position-velocity potential holomorphic coordinates, and prove that small localized data leads to global solutions. This article is a continuation of authors' earlier paper [8].

1. Introduction

We consider the two dimensional infinite depth water wave equations with gravity but without surface tension. This is governed by the incompressible Euler's equations with boundary conditions on the water surface. Under the additional assumption that the flow is irrotational the fluid dynamics can be expressed in terms of a one-dimensional evolution of the water surface coupled with the trace of the velocity potential on the surface.

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This problem was previously considered by many other authors. The local in time existence and uniqueness of solutions was proved in [14, 21, 17], both for finite and infinite depth. Later, Wu [19] proved almost global existence for small localized data. Very recently, global results for small localized data were independently obtained by Alazard-Delort [3] and by Ionescu-Pusateri [11]. Extensive work was also done on the same problem in three or higher space dimensions, and also on related problems with surface tension, vorticity, finite bottom, etc. Without being exhaustive, we list some of the more recent references [2, 1, 5, 6, 7, 12, 4, 13, 15, 16, 18, 20, 22].

An essential choice in any approach to this problem is that of the coordinates used. The citations above largely rely on either Eulerian or Lagrangian coordinates. Instead, the present article relies on holomorphic coordinates, which were originally introduced by Nalimov [14]; these are briefly described below. In the earlier article [8], using holomorphic coordinates, we revisited this problem in order to provide a new, self-contained approach, which considerably simplified and improved on many of the results mentioned above. Our results included:

(i) local well-posedness in Sobolev spaces, improving on previous regularity thresholds, e.g., those in [2].

(ii) cubic lifespan bounds for small data. These are proved using a *modified energy method*, first introduced in the authors' previous article [9]. The idea there is that instead of trying to transform the equation using the normal form method, which does not work well in quasilinear settings, one can produce quasilinear energy functionals, which are conserved to cubic order.

(iii) almost global well-posedness for small localized data, providing a shorter, simpler alternative to Wu's approach in [19], with lower regularity thresholds.

Here we improve the result in (iii) to a global statement, improving and simplifying the earlier results of Alazard-Delort [3] and by Ionescu-Pusateri [11] in several ways; see the discussion following our main theorem.

We first recall the set-up and the equations. We denote the water domain at time t by $\Omega(t)$, and the water surface at time t by $\Gamma(t)$. We think of $\Gamma(t)$ as being asymptotically flat at infinity. Rather than working in cartesian coordinates and the Eulerian setting, we use time dependent coordinates defined via a conformal map $\mathcal{F} : \mathbb{H} \to \Omega(t)$, where $\mathbb{H}$ is the lower half plane, $\mathbb{H} := \{\alpha + i\beta \ : \ \beta < 0\}$. We also have $\mathcal{F}(\mathbf{R}) = \Gamma(t)$. We call these the holomorphic coordinates.

The real variable α is then used to parametrize the free surface $\Gamma(t)$. We say that a function of α is holomorphic if its Fourier transform is supported in $(-\infty, 0]$. They can be described by the relation $Pf = f$, where the projector operator P to negative frequencies can be defined using the Hilbert transform

H as

$$P := \frac{1}{2}(I - iH).$$

Our variables (Z, Q) are functions of t and α which represent the position of the water surface $\Gamma(t)$, respectively the holomorphic extension of the velocity potential restricted to $\Gamma(t)$, expressed in the holomorphic coordinates. In view of our choice of coordinates, it is natural to consider the evolution of (Z, Q) within the closed subspace of holomorphic functions within various Sobolev spaces.

In position-velocity potential holomorphic coordinates the equations have the form

$$\begin{cases} Z_t + FZ_\alpha = 0 \\ Q_t + FQ_\alpha - i(Z - \alpha) + P\left[\dfrac{|Q_\alpha|^2}{J}\right] = 0, \end{cases}$$

where

$$F := P\left[\frac{Q_\alpha - \bar{Q}_\alpha}{J}\right], \qquad J := |Z_\alpha|^2.$$

For the derivation of the above equations we refer the reader to [8], Appendix A. With the substitution $W := Z - \alpha$ they become

$$\begin{cases} W_t + F(1 + W_\alpha) = 0 \\ Q_t + FQ_\alpha - iW + P\left[\dfrac{|Q_\alpha|^2}{J}\right] = 0, \end{cases} \tag{1.1}$$

where

$$F = P\left[\frac{Q_\alpha - \bar{Q}_\alpha}{J}\right], \qquad J = |1 + W_\alpha|^2.$$

We can also differentiate and rewrite the system in terms of the diagonal variables

$$(\mathbf{W}, R) := \left(W_\alpha, \frac{Q_\alpha}{1 + W_\alpha}\right).$$

This yields the self-contained system

$$\begin{cases} \mathbf{W}_t + b\mathbf{W}_\alpha + \dfrac{(1 + \mathbf{W})R_\alpha}{1 + \bar{\mathbf{W}}} = (1 + \mathbf{W})M \\ R_t + bR_\alpha = i\left(\dfrac{\mathbf{W} - a}{1 + \mathbf{W}}\right), \end{cases} \tag{1.2}$$

where the real *advection velocity* b is given by

$$b := P\left[\frac{Q_\alpha}{J}\right] + \bar{P}\left[\frac{\bar{Q}_\alpha}{J}\right],$$

and the real *frequency-shift* a is

$$a := i\left(\bar{P}\left[\bar{R}R_\alpha\right] - P\left[R\bar{R}_\alpha\right]\right).$$