MONODROMIES AT INFINITY OF NON-TAME POLYNOMIALS

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ABSTRACT. — Polynomials that we usually encounter in mathematics are non-convenient and hence non-tame at infinity. We consider the monodromy at infinity and the monodromies around the bifurcation points of polynomial functions $f : \mathbb{C}^n \to \mathbb{C}$ which are non-tame at infinity and might have non-isolated singularities. Our description of their Jordan blocks in terms of the Newton polyhedra and the motivic Milnor fibers relies on two new issues: the non-atypical eigenvalues of the monodromies and the corresponding concentration results for their generalized eigenspaces.

Résumé (Monodromies à l’infini des polynômes non-modérés). — Les polynômes qu’on rencontre d’habitude en mathématiques sont généralement non-commodes et donc non-modérés à l’infini. On considère ici la monodromie à l’infini et les monodromies autour les valeurs de bifurcation des fonctions polynomiales $f : \mathbb{C}^n \to \mathbb{C}$ qui sont non-modérées à l’infini et peuvent avoir des singularités non-isolées. Notre description de leurs blocs de Jordan en termes des polyèdres de Newton et des fibres de Milnor motiviques s’appuie sur deux nouveaux concepts : les valeurs propres non-atypiques des monodromies et les résultats de concentration pour leurs espaces propres généralisés.
1. Introduction

For a polynomial map \( f : \mathbb{C}^n \to \mathbb{C} \), it is well-known that there exists a finite subset \( B \subset \mathbb{C} \) such that the restriction
\[
\mathbb{C}^n \setminus f^{-1}(B) \to \mathbb{C} \setminus B
\]
of \( f \) is a locally trivial fibration. We denote by \( B_f \) the smallest subset \( B \subset \mathbb{C} \) satisfying this condition. We call the elements of \( B_f \) bifurcation points of \( f \).

For \( f(x) = \sum_{v \in \mathbb{Z}^n} a_v x^v \) \((a_v \in \mathbb{C})\) we call the convex hull of \( \text{supp} \ f = \{ v \in \mathbb{R}^n \mid a_v \neq 0 \} \) in \( \mathbb{R}^n \) the Newton polytope of \( f \) and denote it by \( \text{NP}(f) \). After Kushnirenko [11], the convex hull \( \Gamma(\infty)(f) \subset \mathbb{R}^n \) of \( \{0\} \cup \text{NP}(f) \) in \( \mathbb{R}^n \) is called the Newton polyhedron at infinity of \( f \).

**Definition 1.1.** — We say that \( f \) is convenient if \( \Gamma(\infty)(f) \) intersects the positive part of the \( i \)-th axis of \( \mathbb{R}^n \) for any \( 1 \leq i \leq n \).

If \( f \) is convenient and non-degenerate at infinity (see Definition 2.1), then by a result of Broughton [1] it is tame at infinity. In this tame case he proved that one has the concentration
\[
H^j(f^{-1}(R); \mathbb{C}) = 0 \quad (j \neq 0, n-1)
\]
for the generic fiber \( f^{-1}(R) (R \gg 0) \) of \( f \). After this fundamental result many mathematicians studied tame polynomials. However, polynomials that we usually encounter in mathematics are non-convenient and hence non-tame at infinity. According to the fundamental result of Némethi and Zaharia [18], they have a lot of singularities at infinity in general. The study of non-tame polynomials is important for the Jacobian Conjecture since non-tame polynomials are the only interesting objects in the problem. Their study would be useful also in the mirror symmetry, where the Landau-Ginzburg potentials may be non-convenient. Moreover, in what concerns the evaluation of the bifurcation set \( B_f \), non-tame polynomials were studied by many mathematicians and with different methods, in particular by Némethi and Zaharia [18], [31] by using Newton polyhedra. The main reason why non-tame polynomials could not be studied precisely before is that one cannot expect to have the concentration (1.2) for them.

In this paper we overcome this difficulty on non-tame polynomials by improving the above-mentioned result of Broughton [1]. Let \( C_R = \{ x \in \mathbb{C} \mid \|x\| = R \} \) \((R \gg 0)\) be a sufficiently large circle in \( \mathbb{C} \) such that \( B_f \subset \{ x \in \mathbb{C} \mid \|x\| < R \} \). Then by restricting the locally trivial fibration \( \mathbb{C}^n \setminus f^{-1}(B_f) \to \mathbb{C} \setminus B_f \) to \( C_R \) we obtain a geometric monodromy automorphism \( \Phi_f^\infty : f^{-1}(R) \to f^{-1}(R) \) and the linear maps
\[
\Phi_f^j : H^j(f^{-1}(R); \mathbb{C}) \to H^j(f^{-1}(R); \mathbb{C}) \quad (j = 0, 1, \ldots)
\]
associated to it, where the orientation of \( C_R \) is taken to be counter-clockwise as usual. We call \( \Phi_j^\infty \)'s the (cohomological) monodromies at infinity of \( f \). In the last few decades many mathematicians studied \( \Phi_j^\infty \)'s from various points of view. In the tame case, Libgober-Sperber [12] obtained a beautiful formula which expresses the semisimple part (i.e., the eigenvalues) of \( \Phi_{j-1}^\infty \) in terms of the Newton polyhedron at infinity of \( f \) (see [13] for its generalizations). Recently in [14] (see also [6]) the first author proved formulae for its nilpotent part, i.e., its Jordan normal form, by using the motivic Milnor fiber at infinity of \( f \). However, the methods of [12], [13] and [14] etc. do not apply beyond the tame case by the absence of the concentration (1.2) for non-tame polynomials (see [16] for a partial result). In this paper, even for non-tame polynomials we show that the desired cohomological concentration holds for the generalized eigenspaces of \( \Phi_j^\infty \) for "good" eigenvalues associated to \( f \). Then by avoiding the remaining "bad" eigenvalues, we can successfully generalize the results in [12], [13] and [14] etc. to non-tame polynomials and completely determine the Jordan normal forms of \( \Phi_j^\infty \).

**Theorem 1.2.** — Let \( f \in \mathbb{C}[x_1, \ldots, x_n] \) be a non-convenient polynomial such that \( \dim \Gamma_\infty(f) = n \). Assume that \( f \) is non-degenerate at infinity. Then for any non-atypical eigenvalue \( \lambda \notin A_f \) of \( f \) we have the concentration

\[
H^j(f^{-1}(R); \mathbb{C}) \cong 0 \quad (j \neq n - 1)
\]

for the generic fiber \( f^{-1}(R) \subset \mathbb{C}^n \) \((R \gg 0)\) of \( f \).

This theorem allows non-isolated singularities of \( f \) and also the situation where the fibers may have cohomological perturbation "at infinity". Indeed, some of its atypical fibers \( f^{-1}(b) \) \((b \in B_f)\) e.g., \( f^{-1}(0) \) have non-isolated singularities in general. In the "tame" case one has only isolated singularities in \( \mathbb{C}^n \) and either vanishing cycles at infinity do not occur at all or they occur at isolated points only (in the sense of [25], [29]), and then the concentration of cohomology (1.2) follows.

Theorem 1.2 will be proved by refining the proof of Sabbah’s theorem [24, Theorem 13.1] in our situation. More precisely we construct a new compactification \( \overline{X}_\Sigma \) of \( \mathbb{C}^n \) and study the "horizontal" divisors at infinity for \( f \) in \( \overline{X}_\Sigma \setminus \mathbb{C}^n \) very precisely to prove the concentration. With this main result at hand, by using the results in [14, Section 2] we can prove the generalizations of [12], [13] and [14, Theorems 5.9, 5.14 and 5.16] etc. to non-tame polynomials and