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HESSIAN OF THE NATURAL HERMITIAN FORM ON TWISTOR SPACES

BY GUILLAUME DESCHAMPS, NOËL LE DU & CHRISTOPHE MOUROUGANE

ABSTRACT. — We compute the hessian $\mathrm{i}\mathrm{d}'\mathrm{d}''\mathbb{W}$ of the natural Hermitian form $\mathbb{W}$ successively on the Calabi family $\mathbb{T}(M, g, (I, J, K))$ of a hyperkähler manifold $(M, g, (I, J, K))$, on the twistor space $\mathbb{T}(M, g)$ of a 4-dimensional anti-self-dual Riemannian manifold (M, g) and on the twistor space $\mathbb{T}(M, g, D)$ of a quaternionic Kähler manifold (M, g, D). We show a strong convexity property of the component of cycle space of the Calabi family of a hyperkähler manifold, that contains twistor lines. We also prove convexity properties of the 1-cycle space of the twistor space $\mathbb{T}(M, g)$ of a 4-dimensional anti-self-dual Einstein manifold (M, g) of non-positive scalar curvature and of the 1-cycle space of the twistor space $\mathbb{T}(M, g, D)$ of a quaternionic Kähler manifold (M, g, D) of non-positive scalar curvature. We check that no non-Kähler strong Kähler with torsion (KT) manifold occurs as such a twistor space.

RÉSUMÉ (*Hessien de la forme hermitienne naturelle sur des espaces de twisteurs*). — Nous calculons le hessien $\mathrm{i}\mathrm{d}'\mathrm{d}''\mathbb{W}$ de la forme hermitienne naturelle $\mathbb{W}$ successivement sur la famille de Calabi $\mathbb{T}(M, g, (I, J, K))$ d'une variété hyperkählérienne $(M, g, (I, J, K))$, sur l'espace des twisteurs $\mathbb{T}(M, g)$ d'une variété riemannienne (M, g)

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de dimension 4 anti-auto duale et sur l'espace des twisteurs $\mathbb{T}(M,g,D)$ d'une variété quaternionique kähler (M,g,D). Nous montrons une propriété de convexité de la composante de l'espace des cycles de la famille de Calabi d'une variété hyperkählérienne, qui contient les droites twistorielles. Nous montrons aussi des propriétés de convexité de l'espace des 1-cycles de l'espace des twisteurs d'une variété d'Einstein de dimension 4 anti-auto duale à courbure scalaire négative et de l'espace des 1-cycles de l'espace des twisteurs d'une variété quaternionique kähler à courbure scalaire négative. Nous vérifions aussi qu'aucune variété fortement kählerienne avec torsion (KT) non kählérienne n'est obtenue par les constructions précédentes.

1. Introduction

The twistor construction is known to provide examples of manifolds endowed with a natural metric $\mathbb{G}$ and a natural almost complex structure $\mathbb{J}$ that is sometimes integrable and often non-Kähler (see Section 2 for precise definitions). Our aim is to compute the exterior derivative and the Hessian of the natural Hermitian form $\mathbb{W} = \mathbb{G}(\mathbb{J}\cdot,\cdot)$ for different twistor constructions.

We derive, under compactness and non-positive scalar curvature assumption for the base space, a convexity property for the connected components of the 1-cycle space $C_1(\mathbb{T})$ of the twistor space $\mathbb{T}$. The analytic space $C_1(\mathbb{T})$ is the analog in the analytic setting of the Chow scheme in the projective setting; it parametrizes linear combinations (with positive integer coefficients) of irreducible compact analytic sets of dimension 1. The convexity property we prove could be a substitute to the well known compactness of the components of the cycle space of compact Kähler manifolds [19, 12].

The classical twistor construction is for anti-self-dual Riemannian 4-manifolds. We can here in full generality compute the Hessian of the natural Hermitian form (see Theorem 4.11). Under extra assumptions on the base Riemannian manifold, we can study the convexity properties of the cycle space $C_1(\mathbb{T})$.

Theorem A (Corollary 4.13). — *The hessian* $\mathrm{i}d'd''\mathbb{W}$ *of the Hermitian form* $\mathbb{W}$ *on the twistor space* $\mathbb{T} = \mathbb{T}(M,g)$ *of a 4-dimensional Einstein manifold* (M,g) *with non-positive constant scalar curvature* s *is non-negative. If furthermore* M *is compact, the volume function on the* 1*-cycle space* $C_1(\mathbb{T})$ *is a continuous pluri-sub-harmonic exhaustion function.*

A similar construction can be made starting with a higher dimensional Riemannian manifold with quaternionic holonomy. A *quaternionic Kähler manifold* is an oriented complete $4n$-dimensional Riemannian manifold (M,g) whose holonomy group is contained in the product $\mathbb{S}(1)\mathbb{S}(n)$ of quaternionic unitary groups. Such a manifold admits a rank 3 sub-bundle $D \subset \mathrm{End}(TM)$

invariant by the Levi-Civita connection of (M,g), locally spanned by a quaternionic triple $(I,J,K = IJ = -JI)$ of almost complex structures g-orthogonal and compatible with the orientation. One can define *the twistor space* $\pi : \mathbb{T} = \mathbb{T}(M,g,D) \to M$ as the bundle of spheres of radius $\sqrt{2}$ of D. Berger proved that quaternionic Kähler manifolds are Einstein.

In the case of positive scalar curvature, the manifold M is compact and Salamon ([20, Theorem 6.1]) showed that its twistor space admits a Kähler-Einstein metric of positive scalar curvature, that coïncides with the metric $\mathbb{G}$, up to changing the choice for the radius of vertical spheres. In particular, $\mathbb{T}$ is a compact complex manifold with positive first Chern class, that is a Fano manifold. The projection onto the vertical direction gives a contact structure ([20, Theorem 4.3]). By the Kähler property of $\mathbb{T}$, every component of its cycle space is compact.

In the case of negative scalar curvature, the twistor space is a complex contact uniruled manifold. The only known compact examples are locally symmetric. We show in this case that the components of the 1-cycle space are pseudo-convex. More precisely, we find the

THEOREM B (Corollary 5.6). — *The hessian* $\mathrm{i}d'd''\mathbb{W}$ *of the Hermitian form* $\mathbb{W}$ *on the twistor space* $\mathbb{T} = \mathbb{T}(M,g,D)$ *of a quaternionic Kähler* $4n$*-manifold* (M,g,D) *with non-positive constant scalar curvature* s *is semi-positive. If furthermore* M *is compact, the volume function on the* 1*-cycle space is a continuous pluri-sub-harmonic exhaustion function.*

In the case of zero scalar curvature, the manifold M is in fact locally hyperkähler. A *hyperkähler manifold* is an oriented $4n$-dimensional Riemannian manifold (M,g) whose holonomy group is contained in the quaternionic unitary group $\mathbb{S}(n)$. In other words, a hyperkähler manifold is an oriented $4n$-dimensional Riemannian manifold (M,g) endowed with a quaternionic triple of global g-Kähler complex structures I, J and K compatible with the orientation. The corresponding pencil of complex structures $f : \mathbb{T} = \mathbb{T}(M,g,D) \to \mathbb{P}^1$ is called the *Calabi family* of $(M,g,D = (I,J,K))$. In this case, we can relate the non-Kähler feature of $\mathbb{T}(M,g,D)$ with the Kodaira-Spencer class of the pencil f.

THEOREM C (Theorem 3.1). — *Let* $(\theta_1,\ldots,\theta_{4n})$ *be a local orthonormal frame of* TM*. For a vertical vector* $U \in \mathcal{V}_{(m,u)}$*,*

$$d''\mathbb{W}_{(m,u)}(U^h, \mathcal{H}\theta_i^a, \mathcal{H}\theta_j^a) = -2\Omega_u(\kappa_U(\theta_i^a), \kappa_U(\theta_j^a))$$

where $\mathcal{H}$ *is the horizontal lift on* $\mathbb{T}$ *given by the Levi-Civita connection on* (M,g)*,* Ω_u *the holomorphic symplectic* $(2,0)$*-form on* $X_u := f^{-1}(u)$*, and* κ_U *is a closed* $(0,1)$*-form on* X_u *with values in* TX_u *that represents the Kodaira-Spencer class of the family* f *at* $u \in \mathbb{P}^1$ *in the direction* U*.*