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JET SCHEMES OF NORMAL TORIC SURFACES

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ABSTRACT. — For $m \in \mathbb{N}, m \geq 1$, we determine the irreducible components of the $m-th$ jet scheme of a normal toric surface S. We give formulas for the number of these components and their dimensions. This permits to determine the log canonical threshold of a toric surface embedded in an affine space. When m varies, these components give rise to projective systems, to which we associate a weighted oriented graph. We prove that, among toric surfaces, the data of this graph is equivalent to the data of the analytical type of S. Besides, we classify these irreducible components by an integer invariant that we call index of speciality. We prove that for m large enough, the set of components with index of speciality 1, is in $1-1$ with the set of exceptional divisors that appear on the minimal resolution of S.

RÉSUMÉ (*Espaces de jets des surfaces toriques normales*). — Pour $m \in \mathbb{N}$, $m \geq 1$, nous déterminons les composantes irréductibles des espaces de m-jets d'une surface torique normales S. Nous donnons des formules pour le nombre de ces composantes et pour leurs dimensions. Ceci permet de déterminer le seuil log-canonique de la surface S plongée dans un espace affine. Quand m varie, ces composantes donnent lieu à des systèmes projectifs, auxquels nous associons un graphe orienté et pondéré. Nous démontrons que, parmi les surfaces toriques, la donnée de ce graphe est équivalente à la donnée du type analytique de S. De plus, nous classifions ces composantes irréductibles via un invariant qu'on appelle indice de spécialité. Nous démontrons que pour m assez large, l'ensemble des composantes avec un indice de spécialité égal à 1, est en correspondance bijective avec l'ensemble des diviseurs exceptionnels qui apparaissent sur la résolution minimale des singularités de S.

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1. Introduction

Nash has introduced the arc space of a variety X in order to investigate the intrinsic data of the various resolutions of singularities of X. The analogy with p-adic numbers has led Kontsevich [13], Denef and Loeser [3] to invent motivic integration and to introduce several rational series that generalize analogous series in the p-adic context [4]. The geometric counterpart of the theory of motivic integration has been used by Ein, Mustata and others to obtain formulas controlling discrepancies in terms of invariant of jet schemes—these are finite dimensional approximations of the arc space—[21, 6, 7, 8]. Roughly speaking, while we can extract informations about abstract resolutions of singularities from the arc space and vice versa, we can extract informations about embedded resolutions of singularities from the jet schemes and vice versa. This partly explains why the arc space of a toric variety—which has been intensively studied [12, 14, 1, 10, 11]—is well understood. Indeed, we know an equivariant abstract resolution of a toric variety, what permits to understand the action of the arc space of the torus on its arc space [10], but an equivariant embedded resolution is less accessible.

The structure of jet schemes of singular algebraic varieties is complicated; despite that they were the subject of numerous article in the last decade, few is known about their geometry for specific class of singularities, except for the following classes: monomial ideals [9], determinantal varieties [5], plane branches [17], quasi-ordinary singularities [2].

In this article, we study the jet schemes of a normal toric surface singularity. We determine their irreducible components and we give formulas for their number and dimensions. We give here a brief description of the results. The data of a toric surface singularity S is equivalent to the data of a cone $\sigma \subset N = \mathbb{Z}^2$ generated by $(1,0)$ and (p,q) for two coprime numbers $0 < p < q$. Let $q/p = [c_2, \ldots, c_{e-1}]$ be the Hirzebruch-Jung continued fraction expansion (see Section 2.2); the embedding dimension of S is equal to e; the equations defining the embedding of S in $\mathbb{A}^e = \mathrm{Spec}\mathbb{K}[x_1, \ldots, x_e]$ are described in Section 2. Let $m \in \mathbb{N}, m \geq 1$ and let S_m^0 be the space of m-jets centered at the singularity of S (see Section 2.1 for preliminaries on jet schemes). For $i = 2, \ldots, e-1$, $s \in \{1, \ldots, \lceil \frac{m}{2} \rceil\}$(i.e., $m \geq 2s-1 \geq 1$) and $l \in \{s, \ldots, L_{i,m}^s\}$, where

$$L_{i,m}^s := \min\{(c_i - 1)s, (m+1) - s\},$$

we define

$$D_{i,m}^{s,l} := \mathrm{Cont}^s(x_i)_m \cap \mathrm{Cont}^l(x_{i+1})_m,$$

where for $p \in \mathbb{N}$, and $f \in \mathbb{K}[x_1, \ldots, x_e]$,

$$\mathrm{Cont}^p(f)_m = \{\gamma \in S_m \mid \mathrm{ord}_\gamma(f) = p\}.$$

We define $C^{s,l}_{i,m} := \overline{D^{s,l}_{i,m}}$ to be the Zariski closure of $D^{s,l}_{i,m}$. We find in Theorem 4.15 the following.

THEOREM. — *Let $m \in \mathbb{N}$, $m \geq 1$. The irreducible components of S^0_m are $C^{s,L^s_{i,m}}_{e-1,m}$ and the $C^{s,l}_{i,m}$, $i = 2, \dots, e-1$, $s \in \{1, \dots, \lceil \frac{m}{2} \rceil\}$ and $l \in \{s, \dots, L^s_{i,m} - 1\}\}$.*

The formulas that we obtain for the codimensions of the irreducible components of S^0_m (see Proposition 4.11) enable us, by applying Mustata's formula [21], to determine the log canonical threshold of the pair $S \subset \mathbb{A}^e$ (e is the embedding dimension). For $e = 3$, the log canonical threshold is 1. For $e \geq 4$, we find in Corollary 4.27 that

$$\mathrm{lct}(S, \mathbb{A}^e) = \frac{e}{2}.$$

Moreover, making use of the truncation morphisms between the jet schemes, we associate with the irreducible components of S^0_m a graph which is weighted by the codimensions of the irreducible components and the embedding dimension of some of these components. We prove in Corollary 4.25 that the data of this graph is equivalent to the analytical type of the surface. Note that motivic invariants of a toric surface singularity do not determine its analytical type [16, 22].

Finally, we classify the irreducible components by a natural invariant that we call index of speciality; this is the order of contact of the generic point of the component with the maximal ideal defining the singular point of S. We prove that for m large enough, the number of irreducible components of S^0_m is in 1-1 correspondence with the divisors appearing on the minimal abstract resolution of singularities of S. This is to compare with the bijectivness of the Nash map for toric varieties [11]. This is also related to a jet schemes approach to a conjecture of Teissier on toric resolution of singularities [26]. This approach is explained in [19] (see also [15]).

The proof of the main theorem uses heavily the description of the defining equations of the embedding $S \subset \mathbb{A}^e$ ([24, 25]), and some syzygies of these equations that we describe and that are ad hoc to the problem. It also uses known results on the arc space of a toric variety [14, 11],[10] and it is by induction on m and on the embedding dimension e. In particular it uses a kind of approximation of the toric surface S by toric surfaces with smaller embedding dimensions.

Some of the results of this paper were announced in [18].

The structure of the paper is as follows: in section two we present a reminder on jet schemes and on toric surfaces. In section three we study the jet schemes of the A_n singularities. The last section is devoted to the toric surfaces of embedding dimension bigger or equal to four.