Bulletin
de la SOCIÉTÉ MATHÉMATIQUE DE FRANCE

ON THE SPECTRUMS OF ERGODIC SCHRODINGER OPERATORS WITH FINITELY VALUED POTENTIALS

Zhiyuan Zhang

Tome 145
Fascicule 2

2017

SOCIÉTÉ MATHÉMATIQUE DE FRANCE
Publié avec le concours du Centre national de la recherche scientifique
pages 295-304

Le *Bulletin de la Société Mathématique de France* est un périodique trimestriel de la Société Mathématique de France.

Fascicule 2, tome 145, juin 2017

Diffusion

Maison de la SMF - Case 916 - Luminy - 13288 Marseille Cedex 9 - France
`christian.smf@cirm-math.fr`

Hindustan Book Agency
O-131, The Shopping Mall
Arjun Marg, DLF Phase 1
Gurgaon 122002, Haryana
Inde

AMS
P.O. Box 6248
Providence RI 02940
USA
`www.ams.org`

Tarifs

Vente au numéro : 43 € ($ 64)
Abonnement électronique : 135 € ($ 202),
avec supplément papier : Europe 179 €, hors Europe 197 € ($ 296)
Des conditions spéciales sont accordées aux membres de la SMF.

Secrétariat : Nathalie Christiaën

Bulletin de la Société Mathématique de France
Société Mathématique de France
Institut Henri Poincaré, 11, rue Pierre et Marie Curie
75231 Paris Cedex 05, France
Tél : (33) 01 44 27 67 99 • Fax : (33) 01 40 46 90 96
`bullsmf@ihp.fr` • `smf.emath.fr`

ISSN 0037-9484 (print) 2102-622X (electronic)

Directeur de la publication : Stéphane SEURET

Bull. Soc. Math. France
145 (2), 2017, p. 295-304

ON THE SPECTRUMS OF ERGODIC SCHRODINGER OPERATORS WITH FINITELY VALUED POTENTIALS

BY ZHIYUAN ZHANG

ABSTRACT. — In this paper, we show that the Lebesgue measure of the spectrum of ergodic Schrödinger operators with potentials defined by non-constant function over any minimal aperiodic finite subshift tends to zero as the coupling constant tends to infinity. We also obtained a quantitative upper bound for the measure of the spectrum. This follows from a result we proved for ergodic Schrödinger operators with potentials generated by aperiodic subshift under a condition on the recurrence property of the subshift. We also show that such condition is necessary for such result.

RÉSUMÉ (*Sur les spectres des opérateurs de Schrödinger ergodique avec les potentiels des valeurs fini*). — Dans cet article, nous montrons que les mesures Lebesgue des spectres des opérateurs de Schrödinger avec les potentiels qui sont définis par les fonctions non-constantes sur un décalage de type fini, minimal et aperiodique tendent vers zero quand le constant du couplage tend vers l'infini. Ce résultat découle d'un résultat plus général dont nous montrons pour les opérateurs de Schrödinger avec les potentiels qui sont engendrés par un décalage de type fini avec certaine condition sous la récurrence. Nous montrons en même temps que cette condition est nécessaire pour obtenir ce résultat.

Texte reçu le 2 mars 2015, modifié le 8 février 2016, accepté le 23 janvier 2017.

ZHIYUAN ZHANG, Institut de Mathématiques de Jussieu - Paris Rive Gauche, Bâtiment Sophie Germain, Bureau 652, 75205 Paris Cedex 13, France • *E-mail :* `zzzhangzhiyuan@gmail.com`

Mathematical subject classification (2010). — 81Q10, 37B10.

Key words and phrases. — Schrödinger operator, finite shift.

0037-9484/2017/295/$ 5.00
doi:10.24033/bsmf.2738

1. Introduction

This paper is motivated by Simon's subshift conjecture (in [6], see also [2]) and the desire to get a better understanding of recently discovered counter-examples in [1].

Given a finite set $\mathcal{A}$, we define the shift transformation T on $\mathcal{A}^{\mathbb{Z}}$ by $T(\omega)_n = \omega_{n+1}$. Let Ω be a $T-$invariant compact subset of $\mathcal{A}^{\mathbb{Z}}$. Let $\mu \in \mathcal{P}(\Omega)$ be an ergodic $T-$invariant measure. Without loss of generality, in this paper we will always assume that $\Omega = \mathrm{supp}(\mu)$, for otherwise we can replace Ω by $\mathrm{supp}(\mu)$. We will assume that for any $\alpha \in \mathcal{A}$, we have $\mu(\{\omega \mid \omega_0 = \alpha\}) > 0$, for otherwise we can replace $\mathcal{A}$ by one of its subsets. To avoid triviality, a function $v : \mathcal{A} \to \mathbb{R}$ is said to be *admissible* if for any two distinct elements $\alpha, \beta \in \mathcal{A}$, we have $v(\alpha) \neq v(\beta)$. Given any admissible function v, we denote

$$\lambda_v := \min_{\alpha,\beta\in\mathcal{A},\alpha\neq\beta} |v(\alpha) - v(\beta)|.$$

For each $\omega \in \Omega$, we consider the Schrödinger operator H_ω on $\ell^2(\mathbb{Z})$ defined by

$$(H_\omega u)_n = u_{n+1} + u_{n-1} + v(\omega_n)u_n. \tag{1.1}$$

Let Σ_ω denote the spectrum of H_ω. By ergodicity, Σ_ω is the same for μ almost every ω. It is also well-known that when (Ω, T) is minimal, Σ_ω is the same for all $\omega \in \Omega$. In either case, we denote by Σ_v the (almost sure) common spectrum.

Consider an aperiodic minimal subshift as above, the common spectrum was suspected to be of zero Lebesgue measure. For CMV matrices, Barry Simon conjectured the following in [6].

Conjecture 1. — *Given a minimal subshift of Verblunsky coefficients which is not periodic, the common essential support of the associated measures has zero Lebesgue measure.*

There is also a Schrödinger version of the subshift conjecture (see [1]). We state it using our notations.

Conjecture 2. — *Given any admissible $v : \mathcal{A} \to \mathbb{R}$, and a minimal subshift $\Omega \subset \mathcal{A}^{\mathbb{Z}}$ which is not periodic, the associated common spectrum Σ_v has zero Lebesgue measure.*

It has been shown that for strictly ergodic subshifts satisfying the so-called Boshernitzan condition, the Schrödinger operators have zero-measure spectrum for any non-constant potentials [3], and for CMV matrices, one has zero-measure supports [4]. More results on subshifts associated operators can be found in [2].

In the recent work of Avila, Damanik and Zhang [1], the subshift conjecture is shown to be false, for both Schrödinger version and the orginal version for CMV matrices. In fact, the authors proved the following theorem for Schrödinger operators (Theorem 1 in [1]). We rephrase it using our notations.

THEOREM 1. — *Given any integer $p \geq 2$, there is an admissible function $v : \{1, \dots, p\} \to \mathbb{R}$, and a minimal subshift $\Omega \subset \{1, \dots, p\}^{\mathbb{Z}}$ which is not periodic, such that the associated spectrum Σ_v has strictly positive Lebesgue measure.*

They also proved a CMV matrices analog (Theorem 2 in [1]) which disproved the subshift conjecture in its original formulation.

In [1], the authors also proved a positive result which roughly says that when the system endowed with an ergodic invariant measure which is relatively simple, the associated density of states measure is purely singular. The precise conditions are formulated as being "almost surely polynomially transitive" and "almost surely of polynomial complexity". The positive result works for every subshift that is uniformly polynomially transitive and of polynomial complexity (see the remarks after Definition 1,2 in [1]). This theorem can be applied to subshifts generated by translations on tori with Diophantine frequencies, certain skew shifts and interval exchange transformations. Note that this theorem does not imply that the measure of the spectrum is zero.

Given this new phenomenon, namely that the minimal aperiodic subshift generated potentials can have positive-measure spectrum, the following question arises naturally.

QUESTION 1. — *Given a minimal aperiodic subshift and a non-constant potential function, how large can the Lebesgue measure of the spectrum be ?*

This paper is an attempt to study this question. One of our main results is the following.

THEOREM 2. — *Given any integer $p \geq 2$, a minimal aperiodic subshift $\Omega \subset \{1, \dots, p\}^{\mathbb{Z}}$. Then for any $0 < \gamma < 1$ the following is true. For any admissible function $v : \{1, \dots, p\} \to \mathbb{R}$, there exists $C > 0$, such that for any $\lambda > 0$, the Lebesgue measure of $\Sigma_{\lambda v}$ is smaller than $C\lambda^{-\gamma}$.*

We actually proved the following more general result for ergodic Schrödinger operators with subshift-generated potentials:

THEOREM 3. — *Given any integer $p \geq 2$, let μ be an ergodic shift invariant measure on $\{1, \dots, p\}^{\mathbb{Z}}$, such that there exists an integer $K > 0$ satisfying $\mu(\{\omega \mid \omega_0 = \omega_1 = \cdots = \omega_{K-1}\}) = 0$. Then for any $0 < \gamma < 1$, there exists a constant $C > 0$, such that for any admissible function $v : \{1, \dots, p\} \to \mathbb{R}$, we have $Leb(\Sigma_v) < C\lambda_v^{-\gamma}$.*