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MEROMORPHIC QUOTIENTS FOR SOME HOLOMORPHIC G-ACTIONS

by Daniel Barlet

Abstract. — Using mainly tools from previous articles we give necessary and sufficient conditions on the G-orbits' configuration in X in order that a holomorphic action of a connected complex Lie group G on a reduced complex space X admits a *strongly quasi-proper meromorphic quotient.* To show how these conditions can be used, we show, when $G = K.B$ with B a closed connected complex subgroup of G and K a real compact subgroup of G, the existence of a strongly quasi-proper meromorphic quotient for the G-action on X, assuming a slightly stronger condition than the existence of such a quotient for the B-action. We also give a similar result when the connected complex Lie group has the form $G = K.A.K$ where A is a closed connected complex subgroup and K is a compact (real) subgroup.

Résumé (*Quotients méromorphes pour certaines G-actions holomorphes*). — En utilisant les résultats de précédents articles, nous donnons des conditions nécessaires et suffisantes sur la configuration des G-orbites dans X pour que l'action holomorphe d'un groupe de Lie complexe connexe sur un espace complexe réduit X admette un *quotient méromorphe fortement quasi-propre*. Pour illustrer l'intérêt de ces conditions, nous montrons, quand $G = K.B$ où B est un sous-groupe connexe complexe fermé et

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K un sous-groupe compact réel de G, l'existence d'un quotient méromorphe fortement quasi-propre pour l'action de G sur X sous une hypothèse légèrement plus forte que l'existence d'un tel quotient pour l'action de B sur X. Nous donnons également un résultat analogue quand $G = K.A.K$ où A est un sous-groupe complexe fermé et connexe et K un sous-groupe compact réel de G.

1. Introduction

In this article we explain how the tools developed in [9], [1], [2] and [3] can be applied to produce, in suitable cases, a meromorphic quotient of a holomorphic action of a connected complex Lie group G on a reduced complex space X. This uses the notion of *strongly quasi-proper map* introduced in *loc. cit.* and our first goal is to give three hypotheses, called [H.1], [H.2], [H.3], on the G-orbits' configuration in X which are *equivalent* to the existence of a *strongly quasi-proper meromorphic quotient*, notion defined in the Section 1.2.

The proof of this equivalence is the content of Proposition 2.7.1 and Theorem 2.8.1. Then we give a sufficient condition [H.1str], asking the existence of a G-invariant set $\Omega_1 \subset X$ which is dense, Zariski open and "good" for the action, to satisfy the condition [H.1].

Note that the conditions [H.1], [H.2], [H.3] introduced in Section 2.7 only depend on the G-orbits' configuration in X, but the condition [H.1str] depends on the action of G on X itself.

The existence theorem for a strongly quasi-proper meromorphic quotient under our three assumptions is applied to prove the following result:

THEOREM 1.0.1. — *Assume that we have a holomorphic action of a connected complex Lie group G on a reduced complex space X. Assume that $G = K.B$ where K is a compact (real) subgroup of G and B a connected complex closed subgroup of G. Assume that the action of B on X satisfies the condition* [H.1str] *on a G-invariant Zariski open dense subset Ω in X*[1]*, and the conditions* [H.2] *and* [H.3]. *Then the G-action satisfies* [H.1str], [H.2] *and* [H.3]*; so it has a strongly quasi-proper meromorphic quotient.*

A first variant of this result is given by the following theorem.

THEOREM 1.0.2. — *Assume that we have a holomorphic action of a connected complex Lie group G on a reduced complex space X. Assume that $G = K.B$ where K is a compact (real) subgroup of G and B a connected complex closed subgroup of G. Assume that K normalizes B and that the B-action satisfies*

1. This precisely means that there exists a G-invariant dense Zariski open set in X which is a "good open set" for the B-action (see Section 2.5)

the conditions [H.1str], [H.2] *and* [H.3]. *Then the G-action satisfies the conditions* [H.1str], [H.2] *and* [H.3] *and so has a strongly quasi-proper meromorphic quotient.*

Here is a second result obtained by a similar method.

THEOREM 1.0.3. — *Let G be a complex connected Lie group and assume that there exists a closed connected complex subgroup A and a compact (real) subgroup K such That $G = K.A.K$. Assume the we have a completely holomorphic action of G on an irreducible complex space X and that the action of A on X satisfies the following properties:*

i) *The hypothesis* [H.1str] *for the A-action is satisfied on a G-invariant (Zariski good) open set Ω_1 in X.*
ii) *The hypothesis* [H.2] *for the A-action is satisfied on a G-invariant open set $\Omega_0 \subset \Omega_1$ in X.*
iii) *The hypothesis* [H.3] *holds for the A-action.*

Then [H.1str], [H.2] *and* [H.3] *hold for the action of G on X. So there exists a SQP meromorphic quotient of X for the G-action.*

Of course the hypothesis $G = K.A.K$ is more "general" than the case $G = K.B$. But the hypothesis of this last theorem is more restrictive for the action on X of the closed connected complex subgroup A of G: we ask also the G-invariance of the dense open subset Ω_0 of Ω_1 (the open set Ω_0 is defined in the condition [H.2]).

We conclude this article with two results (see Section 3.4) relating the SQP meromorphic quotients for the actions of B and G (resp. of A and G) when they exist:

1. The existence of a holomorphic map $h : Q_B \to Q_G$ (resp. $Q_A \to Q_G$) between the corresponding quotients.
2. The existence under the hypotheses of the Theorem 1.0.1 (resp. the Theorem 1.0.3) of a G-invariant dense Zariski open set Ω disjoint from the centers of the modifications, such that the corresponding map $h_\Omega : q_B(\Omega) \to q_G(\Omega)$ (resp. $h_\Omega : q_A(\Omega) \to q_G(\Omega)$) is proper.

2. Strongly quasi-proper meromorphic quotients

2.1. Preliminaries. — For the definition of the topology on the space $\mathcal{C}_n^f(X)$ of finite type n-cycles in X and its relationship with the topology of the space $\mathcal{C}_n^{\mathrm{loc}}(X)$ we refer to [4] ch. IV, [2] and [3].

For the convenience of the reader we recall shortly here the definitions of a geometrically f-flat map (f-GF map) and of a strongly quasi-proper map (SQP map) between irreducible complex spaces and we give a short summary on some properties of the SQP maps. For more details on these notions see [2] and [3].