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TOPOLOGICAL SUBSTITUTION FOR THE APERIODIC RAUZY FRACTAL TILING

BY NICOLAS BÉDARIDE, ARNAUD HILION & TIMO JOLIVET

ABSTRACT. — We consider two families of planar self-similar tilings of different nature: the tilings consisting of translated copies of the fractal sets defined by an iterated function system, and the tilings obtained as a geometrical realization of a topological substitution (an object of purely combinatorial nature, defined in [6]). We establish a link between the two families in a specific case, by defining an explicit topological substitution and by proving that it generates the same tilings as those associated with the Tribonacci Rauzy fractal.

RÉSUMÉ (*Substitution topologique pour la pavage fractal apériodique de Rauzy*). — On considère deux familles de pavages auto-similaires de nature différente : ceux obtenus par translation de copies d'un ensemble fractal défini par un système de fonctions itérées, et ceux obtenus comme la réalisation géométrique d'une substitution topologique (un objet purement combinatoire, défini dans [6]). On établit un lien entre les deux familles dans un cas particulier, en définissant une substitution topologique explicitement puis en démontrant qu'elle engendre les mêmes pavages que ceux associés au fractal Tribonacci de Rauzy.

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NICOLAS BÉDARIDE, Aix Marseille Univ, CNRS, Centrale Marseille, I2M, Marseille, France. • *E-mail :* nicolas.bedaride@univ-amu.fr
ARNAUD HILION, Aix Marseille Univ, CNRS, Centrale Marseille, I2M, Marseille, France. • *E-mail :* arnaud.hilion@univ-amu.fr
TIMO JOLIVET, Aix Marseille Univ, CNRS, LIF, Marseille, France. • *E-mail :* timo.jolivet@lif.univ-mrs.fr

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1. Introduction

1.1. Main result and motivation. — Self-similar tilings of the plane are characterized by the existence of a common subdivision rule for each tile, such that the tiling obtained by subdivising each tile is the same as the original one, up to a contraction. These tilings have been introduced by Thurston [32] and they are studied in several fields including dynamical systems and theoretical physics, see [5]. A particular class of self-similar tilings arises from *substitutions*, which are "inflation rules" describing how to replace a geometrical shape by a union of other geometrical shapes (within a finite set of basic shapes). Among these, an important class consists of the planar tilings by the so-called Rauzy fractals associated with some one-dimensional substitutions. These fractals are used to provide geometrical interpretations of substitution dynamical systems. They also provide an interesting class of aperiodic self-similar tilings of the plane, see [16, 8].

The aim of this article is to establish a formal link between two self-similar tilings constructed from two different approaches:

- Using an *iterated function system (IFS)*, that is, specifying the shapes and the positions of the tiles with planar set equations (using contracting linear maps), which define the tiles as unions of smaller copies of other tiles. In particular, an IFS does make use of the Euclidean metric of the plane.
- Using a *topological substitution*, that is, specifying which tiles are allowed to be neighbors, and how the neighboring relations are transferred when we "inflate" the tiles by substitution to construct the tiling. With this kind of substitution, there is no use in anyway of a the Euclidean metric: the tiles do not have a metric shape (they are just topological disks).

In other words, we tackle the following question:

> *Given a tiling defined by an IFS, is there a topological substitution which generates an equivalent tiling? If yes, how can we construct it? In other words, when is it possible to describe the geometry of a self-similar tiling (geometrical constraints) by using a purely combinatorial rule (combinatorial constraints) ?*

In this article we answer this question for the tilings of the plane by translated copies of the Rauzy fractals associated with the Tribonacci substitution (which are defined by an IFS). We define a particular topological substitution σ (Figure 3.3, p. 588) and we prove that the Tribonacci fractal tiling $\mathcal{T}_{\mathsf{frac}}$ and the tiling $\mathcal{T}_{\mathsf{top}}$ generated by the topological substitution are equivalent in a strong way. More precisely:

- Associated with the Tribonacci substitution $s : 1 \mapsto 12,\ 2 \mapsto 13,\ 3 \mapsto 1$, there is a dual substitution $\mathbf{E}$ (see Section 4.2) which acts on facets in $\mathbb{R}^3$. Iteration of this dual substitution gives rise to a stepped surface Σ_{step} (a surface which is a union of facets), that is included in the 1-neighborhood of some (linear) plane $\mathcal{P}$ in $\mathbb{R}^3$. Projecting the stepped surface Σ_{step} (and its facets) on $\mathcal{P}$ gives rise to a tiling $\mathcal{T}_{\mathsf{step}}$ of $\mathcal{P}$. It is known [2, 8] that the tiling $\mathcal{T}_{\mathsf{frac}}$ is strongly related to a tiling $\mathcal{T}_{\mathsf{step}}$.

- The topological substitution σ can be iterated on a tile C, giving rises to a 2-dimensional CW-complex $\sigma^\infty(C)$ homeomorphic to a plane, see Section 3.2. However, this complex is not embedded a priori in a plane, even if it turns out that $\sigma^\infty(C)$ can be effectively realized as a tiling $\mathcal{T}_{\mathsf{top}}$ of the plane, see Proposition 3.11. To locate a tile T in $\sigma^\infty(C)$ relatively to another one T', we build a vector (an *"position"*) $\omega_0(T, T') \in \mathbb{Z}^3$: by construction, this vector depends a priori on the choice of a combinatorial path from T to T' in $\sigma^\infty(C)$, and we have to prove that in fact it is independent of the path, see Section 5.1.
- Since it is already explained in the literature how to relate $\mathcal{T}_{\mathsf{frac}}$ and Σ_{step}, and since we explain how $\mathcal{T}_{\mathsf{top}}$ is build from $\sigma^\infty(C)$, the main result of the paper is Theorem 5.16 that states an explicit formula which define a bijection Ψ between tiles in $\sigma^\infty(C)$ and facets in Σ_{step}: we reproduce it just below.

THEOREM. — The map Ψ defined, for every tile T of $\sigma^\infty(C)$, by:

$$\Psi(T) = [\mathbf{M}_s^3(\omega_0(T, C) + \mathbf{u}_{\mathrm{type}(T)}), \theta(\mathrm{type}(T))]^* \tag{1.1}$$

is a bijection from the set of tiles of $\sigma^\infty(C)$ to the set of facets of Σ_{step}.

The notation used to state this theorem will be introduced along the paper. But we want to stress that the fact the formula (1.1) makes use of the position map ω_0 ensures that if two tiles T and T' are close in $\sigma^\infty(C)$, then their images $\Psi(T)$ and $\Psi(T')$ will be close in Σ_{step}. In fact, it is easy to convince oneself that something like that should be true by having a look at Figure 1.1, where three corresponding subsets of the tilings $\mathcal{T}_{\mathsf{top}}$, $\mathcal{T}_{\mathsf{frac}}$ and $\mathcal{T}_{\mathsf{step}}$ are given.

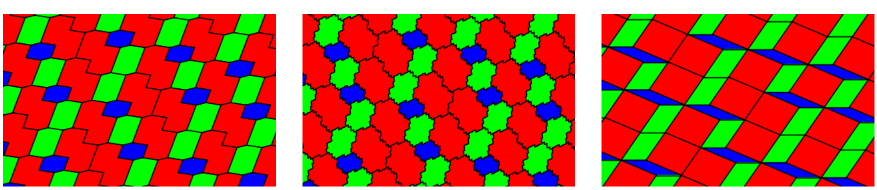

FIGURE 1.1. The three tilings $\mathcal{T}_{\mathsf{top}}$, $\mathcal{T}_{\mathsf{frac}}$ and $\mathcal{T}_{\mathsf{step}}$ (from left to right).

On Figure 1.1, it is also worth to notice that the underlying CW-complexes of $\mathcal{T}_{\mathsf{top}}$ and $\mathcal{T}_{\mathsf{step}}$ are not the same. Indeed, the valence of a vertex in $\mathcal{T}_{\mathsf{top}}$ is either 2 or 3, whereas the valence of a vertex in $\mathcal{T}_{\mathsf{step}}$ can be equal to 3, 4, 5 or 6. In that sense, the two tilings $\mathcal{T}_{\mathsf{step}}$ and $\mathcal{T}_{\mathsf{top}}$ are really different.

We have chosen to present our results on a specific substitution rather than in a general form because it makes presentation clearer and it avoids many "artificial" technicalities. Moreover, we do not know what a general answer to the above question may look like. However, we give some insight about this general question in Section 6.

1.2. Comparison of some different notions of substitutions. — The word "substitution" is used in many different ways in the literature. The list below reviews several such notions, going from the most geometrical one (IFS) to the most combinatorial one (topological substitutions). Indeed, as observed by Peyrière [26], having a combinatorial description of substitutive tiling turns out to be very useful in many situations. This list is not exhaustive, it only contains the notions of substitutions that we explicitly use in this article. See [18] for another survey about geometrical substitutions.

One-dimensional symbolic substitutions. — These substitutions are used to generated infinite one-dimensional words which are studied mostly for their word-theoretical and dynamical properties. An example is the Tribonacci substitution $1 \mapsto 12, 2 \mapsto 13, 3 \mapsto 1$ defined in Section 4.3. See [16] for a classical reference. This is the only notion of the present list which is only symbolic (not geometrical).

Self-affine substitutions (iterated function systems). — Also known as *substitution Delone sets* [24], this notion is a particular class of iterated functions systems, where it is required that the geometrical objects defined by the IFS are compact sets which are the closure of their interior, in such a way that tilings can be defined. See Proposition 4.5 for an example of such a definition for the Tribonacci fractal.

Dual (or "generalized") substitutions. — These substitutions, introduced in [4] can be seen as a discrete version of self-affine substitutions. Instead of defining fractal tilings in a purely geometrical way (like with IFS), these substitutions act on unions of faces of unit cubes located at integer coordinates. We define the associated fractal sets and tilings by iterating the dual substitution and by taking a Hausdorff limit of the (renormalized) unions of unit cube faces. The fact that we deal with unit cube faces allows us to exploit some fine combinatorial and topological properties of the resulting patterns. This provides some powerful tools in the study of substitution dynamics and Rauzy fractal topology. Dual substitutions are usually denoted by $\mathbf{E}_1^*(\sigma)$, where σ is a one-dimensional symbolic substitution, See [8, 30] for many references and results, and Definition 4.2 for the particular example studied in this article.

Local substitution rules. — This notion has been used to tackle combinatorial questions about substitution dynamics [22, 2, 3, 7] and have also been studied in a more general context [14, 23]. Their aim is to get a "more combinatorial" version of dual substitutions. Instead of computing explicitely the coordinates of the image of each unit cube face (like we do for dual substitutions), we give some *local rules* (or *concatenation* rules) for "gluing together" the images of two adjacent faces. The map defined in Figure 5.2, p. 601 is an example of such a substitution (except that it is defined over topological tiles and not unit cube faces).

Topological substitutions. — Introduced in [6], topological substitutions do not make any use of geometry: the tiles are topological disks (with no Euclidean shape), the boundaries of which have a simplicial structure (made of vertices and edges). It is a notion less geometrically rigid than the previous ones. They act on CW-complexes, and the "gluing rules" are more abstract and combinatorial than local substitution rules. A topological substitution generates a CW-complex homeomorphic to the plane. If this complex can be geometrized as a tiling of the plane, we say that the tiling is a topological substitutive tiling. Topological substitutions allowed for instance to prove that there is no substitutive primitive tiling of the hyperbolic plane, even though an explicit example of a non-primitive topological substitution which generates a tiling of the hyperbolic plane is given in [6].

In order to distinguish this notion of substitution used in the present article from the other combinatorial notions discussed in this introduction, we use the term *topological substitution* instead of *combinatorial substitution*

The examples of topological substitutions given in the present article (Figure 3.3 and Figure 6.1) are interesting because they provide new examples of topological substitutive tilings, which can be realized as (substitutive) tilings of the plane.

Other related notions. — There is another notion, elaborated by Fernique and Ollinger [15] (and developped in details in the specific case of Tribonacci), which lies between local substitution rules and topological substitutions. For these so-called *combinatorial substitutions*, the Euclidean shape of the tiles is specified, and the matching rules are stated in terms of colors associated with some subintervals on the boundaries of the tiles and their images. We stress that, in that case, the Euclidean geometry is used both to give the shape of the tiles and to specify that two tiles with same shape differ with a translation of the plane.

Purely combinatorial notions of substitutions have already been defined. For instance, Priebe-Frank [17] introduced a very natural notion of (labeled) graph substitutions. In the case of a substitutive tiling, this graph substitution has to be understood as a substitution on the dual graph to the tiling. The main issue with this formalism is that there is no a priori control on the planarity of the graph obtained by iteration of the substitution, and thus in general the limit graph obtained by iteration can not be the dual graph to any tiling of the plane. Topological substitutions of [6] remedy this problem.