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TORSION AND SYMPLECTIC VOLUME IN SEIFERT MANIFOLDS

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ABSTRACT. — For any oriented Seifert manifold X and compact connected Lie group G with finite center, we relate the Reidemeister density of the moduli space of representations of the fundamental group of X into G to the Liouville measure of some moduli spaces of representations of surface groups into G.

RÉSUMÉ (*Torsion et volume symplectique des variétés de Seifert*). — Pour toute variété de Seifert orientée X et tout groupe de Lie compact connexe G de centre fini, nous calculons la densité de Reidemeister de l'espace des modules des représentations du groupe fondamental de X dans G en fonction de la mesure de Liouville de certains espaces de modules de représentations de groupes de surfaces.

1. Introduction

For any Lie group G and manifold Y, the moduli space $\mathcal{M}(Y)$ of conjugacy classes of representations of $\pi_1(Y)$ in G, has natural differential geometric structures. If Σ is a closed oriented surface, $\mathcal{M}(\Sigma)$ has a symplectic stucture

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defined via intersection pairing [1], [4]. More generally, if Σ is a compact oriented surface and $u \in \mathcal{M}(\partial\Sigma)$, the subspace $\mathcal{M}(\Sigma, u)$ of $\mathcal{M}(\Sigma)$ consisting of the representations restricting to u on the boundary has a natural symplectic structure. If X is a closed 3-dimensional oriented manifold, $\mathcal{M}(X)$ has a natural density μ_X defined from Reidemeister torsion [16].

In this article, we relate these structures for X any oriented Seifert manifold and Σ a convenient oriented surface embedded in X. We will prove that when G is compact with finite center, the subspace $\mathcal{M}^0(X) \subset \mathcal{M}(X)$ of irreducible representations, is a smooth manifold covered by disjoint open subsets O_α, such that each O_α identifies with $\mathcal{M}^0(\Sigma, u_\alpha)$ for some $u_\alpha \in \mathcal{M}(\partial\Sigma)$. Furthermore, on each U_α the canonical density μ_X identifies, up to some multiplicative constant depending on α, with the Liouville measure of the symplectic structure of $\mathcal{M}^0(\Sigma, u_\alpha)$.

Our main motivation is the Witten's asymptotic conjecture, which predicts that the Witten-Reshetikhin-Turaev invariant of a 3-manifold X has a precise asymptotic expansion in the large level limit. This expansion is a sum of oscilatory terms, whose amplitudes are function of the Reidemeister volume of the components of $\mathcal{M}(X)$. In the case where X is a Seifert manifold, some of these amplitudes are actually function of the symplectic volumes of the moduli spaces $\mathcal{M}^0(\Sigma, u)$, [13], [3]. So a relation between Reidemeister and symplectic volumes was expected. At a more general level, it is known that the Chern-Simons theory on a Seifert manifold can be interpreted as two-dimensional Yang-Mills theory [2].

Let us state our results with more detail and then discuss the related literature.

Statement of the main result. — The Seifert manifolds we will consider are the oriented closed connected three manifold equipped with a locally free circle action. Any such manifold may be obtained as follows. Let Σ be an oriented compact surface with $n \geqslant 1$ boundary components $C_1, \ldots, C_n$. Let D be the standard closed disk of $\mathbb{C}$. Let φ_i be an orientation reversing diffeomorphism from $\partial D \times S^1$ to $C_i \times S^1$. Let X be the manifold obtained by gluing n copies of $D \times S^1$ to $\Sigma \times S^1$ through the maps φ_i. We have $[\varphi_i(\partial D)] = -p_i[C_i] + q_i[S^1]$ in $H_1(C_i \times S^1)$ where p_i, q_i are two relatively prime integers. We assume that $p_i \geqslant 1$ for all i.

Let G be a compact connected Lie group with finite center. For $Y = X$, Σ, C_i or S^1, we denote by $\mathcal{M}(Y)$ (resp. $\mathcal{M}^0(Y)$) the set of representations (resp. irreducible representations) of $\pi_1(Y)$ in G up to conjugation. Since C_i and S^1 are oriented circles, we can identify $\mathcal{M}(C_i)$ and $\mathcal{M}(S^1)$ with the set $\mathcal{C}(G)$ of conjugacy classes of G. For any $u \in \mathcal{C}(G)^n$, we denote by $\mathcal{M}^0(\Sigma, u)$ the subset of $\mathcal{M}^0(\Sigma)$ consisting of the representations whose restriction to each C_i is u_i. Recall that $\mathcal{M}^0(\Sigma, u)$ is a smooth symplectic manifold.

For any $(u,v) \in \mathcal{C}(G)^{n+1}$, we denote $\mathcal{M}^0(X,u,v)$ the subset of $\mathcal{M}^0(X)$ consisting of representations whose restriction to each C_i is u_i and to S^1 is v. Let $\mathcal{P}$ be the subset of $\mathcal{C}(G)^{n+1}$ consisting of the (u,v) such that $\mathcal{M}^0(X,u,v)$ is non empty.

THEOREM 1.1. — *$\mathcal{M}^0(X)$ is a smooth manifold, whose components may have different dimensions. For any $[\rho] \in \mathcal{M}^0(X)$, the tangent space $T_{[\rho]}\mathcal{M}^0(X)$ is canonically identified with $H^1(X, \operatorname{Ad}\rho)$ where $\operatorname{Ad}\rho$ is the flat vector bundle associated to ρ via the adjoint representation. Furthermore, $\mathcal{P}$ is finite and for any $(u,v) \in \mathcal{P}$, $\mathcal{M}^0(X,u,v)$ is an open subset of $\mathcal{M}^0(X)$ and the restriction map $R_{u.v}$ from $\mathcal{M}^0(X,u,v)$ to $\mathcal{M}^0(\Sigma,u)$ is a diffeomorphism.*

For any irreducible representation ρ of $\pi_1(X)$ in G, the homology groups $H_0(X, \operatorname{Ad}\rho)$ and $H_3(X, \operatorname{Ad}\rho)$ are trivial. By Poincaré duality, $H_2(X, \operatorname{Ad}\rho)$ is the dual of $H_1(X, \operatorname{Ad}\rho)$. So the Reidemeister torsion of $\operatorname{Ad}\rho$ is a non vanishing element of $\big(\det H_1(X, \operatorname{Ad}\rho)\big)^{-2}$ well-defined up to sign. Consequently, the inverse of the square root of the torsion is a density of $H^1(X, \operatorname{Ad}\rho)$. Since $H^1(X, \operatorname{Ad}\rho)$ identifies with the tangent space of $\mathcal{M}^0(X)$ at ρ, we define in this way a density μ_X on $\mathcal{M}^0(X)$.

For any $u \in \mathcal{C}(G)$ and $[\rho] \in \mathcal{M}^0(\Sigma,u)$, the tangent space $T_{[\rho]}\mathcal{M}^0(\Sigma,u)$ is identified with the kernel of the morphism $H^1(\Sigma, \operatorname{Ad}\rho) \to H^1(\partial\Sigma, \operatorname{Ad}\rho)$. The symplectic product of $T_{[\rho]}\mathcal{M}^0(\Sigma,u)$ is induced by the intersection product of $H^1(\Sigma, \operatorname{Ad}\rho)$ with $H^1(\Sigma, \partial\Sigma, \operatorname{Ad}\rho)$. We denote by μ_u the corresponding Liouville measure of $\mathcal{M}^0(\Sigma,u)$.

As a last definition, let $\Delta : \mathcal{C}(G) \to \mathbb{R}$ be the function given by

$$\Delta(u) = \big|\det_{H_g}(\operatorname{Ad}_g - \operatorname{id})\big|^{1/2},$$

where g is any element in the conjugacy class u and H_g is the orthocomplement of $\ker(\operatorname{Ad}_g - \operatorname{id})$. Equivalently, let $\mathfrak{t}$ be the Lie algebra of a maximal torus of G, $R \subset \mathfrak{t}^*$ be the corresponding set of real roots and $R_+ \subset R$ be a set of positive roots. Then for any $X \in \mathfrak{t}$,

$$\Delta([e^X]) = \prod_{\alpha \in R_+;\ \alpha(X) \neq 0} 2|\sin(\pi\alpha(X))|.$$

THEOREM 1.2. — *For any $(u,v) \in \mathcal{P}$, we have on $\mathcal{M}^0(X,u,v)$*

$$\mu_X = \left(\prod_{i=1}^{n} \frac{\Delta(u_i^{r_i})}{p_i^{(\dim G - \dim u_i)/2}}\right) R^*_{u,v}\mu_u,$$

where $R_{u,v}$ is the restriction map from $\mathcal{M}^0(X,u,v)$ to $\mathcal{M}^0(\Sigma,u)$ and for each i, r_i is any inverse of q_i modulo p_i, and $u_i^{r_i} \in \mathcal{C}(G)$ is the conjugacy class containing the g^{r_i} for $g \in u_i$.

Several definitions require an invariant scalar product on the Lie algebra of G: the symplectic structure of $\mathcal{M}^0(\Sigma, u)$, the Poincaré duality between $H_1(X, \operatorname{Ad}\rho)$ and $H_2(X, \operatorname{Ad}\rho)$ and the Reidemeister torsion of $\operatorname{Ad}\rho$. Our implicit convention is to choose the same invariant scalar product each time.

During the proof, we will prove interesting intermediate results:

- for any irreducible representation ρ of $\pi_1(X)$ in G, the cohomology groups $H^1(X, \operatorname{Ad}\rho)$ and $H^2(X, \operatorname{Ad}\rho)$ both identify naturally with the kernel of the restriction morphism $H^1(\Sigma, \operatorname{Ad}\rho) \to H^1(\partial\Sigma, \operatorname{Ad}\rho)$.
- by these identifications, the intersection product of $H^1(X, \operatorname{Ad}\rho)$ with $H^2(X, \operatorname{Ad}\rho)$ is sent to the intersection product of $H^1(\Sigma, \operatorname{Ad}\rho)$ with $H^1(\Sigma, \partial\Sigma, \operatorname{Ad}\rho)$.
- the Reidemeister torsion of $\operatorname{Ad}\rho \to X$ is equal to $C^{-2} \det \psi$ where $\psi : H_1(X, \operatorname{Ad}\rho) \to H_2(X, \operatorname{Ad}\rho)$ is the map induced by the previous identifications and C is the factor appearing in Theorem 1.2.

This results are respectively proved in Sections 4, 5 and 6. Theorem 1.2 is proved in Section 7 and Theorem 1.1 in Section 3.2.

Related results in the litterature. — Witten [17] proved that for S a closed oriented surface, the canonical density μ_S of $\mathcal{M}^0(S)$ defined from Reidemeister torsion, is the Liouville measure of the natural symplectic structure of $\mathcal{M}^0(S)$. He also extended this result to surfaces with boundary. We tried to deduce Theorem 1.2 from this by relating the torsions of $\operatorname{Ad}\rho \to X$ and $\operatorname{Ad}\rho \to \Sigma$, without any success. Our actual proof does not use Witten's result.

Witten also computed explicitely the volumes $\int_{\mathcal{M}^0(\Sigma,u)} \mu_u$, cf. [17], Formula 4.114. For $G = \mathrm{SU}(2)$ and non central conjugacy classes u_i, Park [12] adapted the Witten's method to compute $\int_{\mathcal{M}^0(X,u,v)} \mu_X$, X being our Seifert manifold. Computing the volume of $\mathcal{M}^0(X, u, v)$ with Theorem 1.2 and Witten's formula, we can extend Park's result to any compact connected Lie group G with finite center and any conjugacy classes u_i.

McLellan [8] proved a result similar to Theorem 1.2 for $G = \mathrm{U}(1)$. To do this, he introduced a Sasakian structure on X and used a computation of the corresponding analytic torsion [14]. We will explain in Section 8 how we can recover McLellan's result by adapting our method, providing an elementary proof.

2. The Seifert manifold X

Let $g, n, p_1, q_1, \dots, p_n, q_n$ be integers such that

$$(1) \qquad g \geqslant 0, \quad n \geqslant 1 \quad \text{and} \quad \forall i, \quad p_i, q_i \text{ are coprime and } p_i \geqslant 1.$$

To such a familly we associate the following manifold X. Let Σ be a compact oriented surface with genus g and n boundary components denoted by $C_1, \dots, C_n$.

Let D be a closed disk and for any i, let $\varphi_i : \partial D \times S^1 \to C_i \times S^1$ be an orientation reversing diffeomorphism such that we have in $H_1(S^1 \times C_i)$,

$$[\varphi_i(\partial D)] = -p_i[C_i] + q_i[S^1], \tag{2}$$

where ∂D and C_i are oriented as boundaries of D and Σ respectively. Then X is obtained by gluing n copies of $D \times S^1$ to $\Sigma \times S^1$ along its boundary through the maps φ_i,

$$X = (\Sigma \times S^1) \cup_{\varphi_1 \cup \cdots \cup \varphi_n} (D \times S^1)^{\cup n}. \tag{3}$$

By construction $\Sigma \times S^1$ is a submanifold of X. In the sequel we often consider Σ and S^1 as submanifolds of X by identifying Σ with $\Sigma \times \{y\}$ and S^1 with $\{x\} \times S^1$, where x and y are some fixed points of Σ and S^1 respectively.

The above definitions are all what we need for this article. Nevertheless, it is interesting to understand this in the context of Seifert manifolds. First, if X is obtained as previously, we can extend the S^1-action on $\Sigma \times S^1$ to X, so that for any i, the action on the i-th copy of $D \times S^1$ is free if $p_i = 1$ and otherwise it has one exceptional orbit with isotropy $\mathbb{Z}_{p_i}$. Conversely, consider any three dimensional closed connected oriented manifold Y equipped with an effective locally free action of S^1. Then choose $n \geqslant 1$ orbits $O_1, \ldots, O_n$ of Y including all the exceptional ones. Let $T_1, \cdots, T_n$ be disjoint saturated open tubular neighborhoods of the $O_1, \ldots, O_n$ respectively. Let Σ be any cross-section of the action on $Y \setminus (T_1 \cup \cdots \cup T_n)$. For any i, set $C_i = (\partial \Sigma) \cap \overline{T}_i$ and define p_i as the order of the isotropy group of O_i and q_i so that $[C_i] = q_i[O_i]$ in $H_1(\overline{T}_i)$, where C_i is oriented as the boundary of Σ and O_i by the S^1-action. Let X be any manifold associated to the data Σ, $(p_1, q_1), \ldots, (p_n, q_n)$ as in (3). Then Y is diffeomorphic to X, cf. [5], Theorem 1.5 or the Section 1 of [10] for more details. We can even choose the diffeomorphism between Y and X so that it commutes with the S^1-action and fixes Σ. The collection

$$(g; (p_1, q_1), \ldots, (p_n, q_n))$$

is called the unnormalized Seifert invariant of Y.

3. Character space of a Seifert manifold

Notations. — Let G be a Lie group. For any connected topological space Y, we denote by $\mathcal{M}(Y)$ the set of conjugacy classes of representations of $\pi_1(Y)$ into G[1]. A representation $\rho : \pi_1(Y) \to G$ is said to be irreducible if the centraliser of $\rho(\pi_1(Y))$ is reduced to the center of G. We denote by $\mathcal{M}^0(Y)$ the subset of $\mathcal{M}(Y)$ consisting of conjugacy classes of irreducible representations.

1. A representation of $\pi_1(Y)$ into G is a group morphism from $\pi_1(Y)$ to G. Two representations ρ, ρ' are conjugate if there exists $g \in G$, such that $\rho'(h) = g\rho(h)g^{-1}$, $\forall h \in G$.