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VARIATIONS ON GROMOV'S OPEN-DENSE ORBIT THEOREM

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ABSTRACT. — We investigate under which conditions a geometric structure which is locally homogeneous on a dense open set is locally homogeneous everywhere. In the case of a 3-dimensional Lorentz metric, this allows us to sharpen the conclusions in Gromov's open-dense orbit theorem.

RÉSUMÉ (*Variations sur le théorème de l'orbite dense-ouverte de M. Gromov*). — Nous étudions sous quelles conditions une structure géométrique qui est localement homogène sur un ouvert dense est localement homogène partout. Dans la cadre des variétés lorentziennes de dimension 3, cela conduit à un renforcement des conclusions dans le théorème de l'orbite dense-ouverte de Gromov.

1. Introduction

The main motivation for this article comes from the following result of M. Gromov, often quoted in the literature as the *open-dense orbit theorem.*

THEOREM 1.1 ([13], Th. 3.3.A). — *Let M be a smooth manifold and $\mathcal{S}$ a smooth rigid geometric structure of algebraic type on M. If the automorphism*

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group of $(M, \mathcal{S})$ *has a dense orbit, then the structure* $\mathcal{S}$ *is locally homogeneous on a dense open subset of* M*.*

Recall that a structure is locally homogeneous if, given any pair of points (x, y) of M, there exists a local isometry f (namely a local diffeomorphism preserving $\mathcal{S}$), defined from a neighborhood of x to a neighborhood of y, and satisfying $f(x) = y$.

The notion of rigid geometric structure of algebraic type was introduced in [13]. It covers a wide range of structures, important examples of which are pseudo-Riemannian metrics on manifolds or affine connections.

Theorem 1.1 actually holds under the weaker assumption that the pseudo-group of local isometries has a dense orbit. It comes as a corollary of a more general result, also proved in [13], stating that for a rigid geometric structure of algebraic type, there exists a dense open subset where the orbits of the pseudo-group of local isometries are closed submanifolds. It is thus clear that whenever one of these orbits is dense, it must be open. The reader wanting to learn more about Gromov's theory of rigid transformation groups will find details in [13, 10, 1, 22].

A beautiful application of Theorem 1.1 can be found in [3], where the authors use Gromov's result to get a full classification of contact Anosov flows on compact manifolds, admitting smooth stable and unstable distributions. Their strategy is to show that such contact flows preserve a smooth pseudo-Riemannian metric, which turns out to be locally homogeneous because of Gromov's theorem and the Anosov dynamics. The end (actually the main part) of the proof consists in classifying the possible algebraic local models. More generally, Theorem 1.1 seems to be a key ingredient in classifying rigid geometric structures of a certain type, with a topologically transitive group of isometries. There is, however, a restriction: the local homogeneity ensured by the theorem is only available on a dense open subset of the manifold, while we would like such a result to hold on the whole manifold. This naturally raises the following question:

Question 1.2. — *Can the maximal open set of local homogeneity given by Theorem 1.1 be a strict open subset of* M*?*

While it is expected that the answer to the previous question should be negative, there are very few instances where one can prove it (see [3] and [7] for nontrivial examples where the authors show local homogeneity everywhere).

1.1. Open-dense orbit theorem and 3-dimensional Lorentz metrics. — If we restrict our attention to pseudo-Riemannian structures, the situation seems to be the following. The answer to Question 1.2 is negative for Riemannian manifolds and pseudo-Riemannian surfaces. In the first case, it is almost obvious, and in the second, we see that whenever the isometry group has a dense orbit, the sectional curvature must be constant, which implies local homogeneity.

One aim of this paper is to study the first nontrivial case in addition to the two preceding ones, namely that of 3-dimensional Lorentz manifolds. Our main result is:

THEOREM A. — *Let (M^3, g) be a smooth closed 3-dimensional Lorentz manifold. If the isometry group* $\mathrm{Iso}(M, g)$ *has a dense orbit, then (M^3, g) is locally homogeneous.*

Observe that we don't make any *a priori* assumption on the group $\mathrm{Iso}(M, g)$. In particular, it might be a discrete group.

Under stronger assumptions, Theorem A can be deduced from previous works. For instance, if we assume that the metric g is real analytic and the manifold M is compact, then S. Dumitrescu showed in [6] that the existence of a nonempty open orbit for the pseudo-group of local isometries led to local homogeneity.

In the smooth category, under the stronger assumption that there exists a 1-parameter flow of isometries with a dense orbit, Theorem A can be derived from [21], where A. Zeghib classifies completely all Lorentzian flows on compact 3-manifolds, which are not equicontinuous.

Let us finally mention that obtaining a generalization of Theorem A to Lorentz manifolds of arbitrary dimension, or to general pseudo-Riemannian structures, seems to be rather challenging. Good examples of topologically transitive pseudo-Riemannian flows, illustrating the conclusions of Theorem 1.1, can be built as follows. Let G be a noncompact simple (or semi-simple) Lie group, and let Γ be a uniform lattice. Let $\{g^t\}$ be a 1-parameter subgroup of G, with noncompact closure in G. It follows from Moore's theorem that $\{g^t\}$ acts ergodically on G/Γ. Let κ_0 be the Killing form on $\mathfrak{g}$. This is a pseudo-Riemannian scalar product, which is $\mathrm{Ad}(g^t)$-invariant. Pushing this scalar product by right translations, one gets a bi-invariant pseudo-Riemannian metric on G, which in turns induces a g^t-invariant metric h_0 on G/Γ. This Killing metric h_0 is actually G-invariant, hence homogeneous. The point is that for many 1-parameter groups $\{g^t\}$ (for instance when $\{g^t\}$ is in a Cartan subgroup A, or when $\{g^t\}$ is unipotent), there are many $\mathrm{Ad}(g^t)$-invariant pseudo-Riemannian scalar products κ on $\mathfrak{g}$ in addition to the Killing form. Actually, one can choose some κs that are $\mathrm{Ad}(g^t)$-invariant, without being $\mathrm{Ad}(H)$-invariant for $\{g^t\} \subsetneq H$. By the same construction as above, κ yields a pseudo-Riemannian metric h on G/Γ for which the isometry group reduces to $\{g^t\}$ (maybe up to finite index). Such a metric is of course no longer homogeneous, but is still locally homogeneous. Now, even for those concrete examples, and apart from particular choices of groups $\{g^t\}$, it does not seem obvious to show that *all* pseudo-Riemannian metrics on G/Γ which are $\{g^t\}$-invariant are locally homogeneous (a generalization of the results of [14] might be useful in this regard).

1.2. Quasihomogeneity. — Let us now discuss purely local problems related to Question 1.2.

We recall that a local Killing field for a geometric structure $\mathcal{S}$ on a manifold M is a vector field X defined on some open set $U \subset M$ and is such that the local flow φ_X^t preserves $\mathcal{S}$. The set of Killing fields defined on U is a Lie subalgebra of the vector fields on U, which we denote as $\mathfrak{kill}(U)$. When the structure $\mathcal{S}$ is rigid, then $\mathfrak{kill}(U)$ is always finite dimensional. It then follows that if $(U_i)_{i\in\mathbb{N}}$ is a nested family of open sets containing a point x, and satisfying $\bigcap_{i\in\mathbb{N}} U_i = \{x\}$, then the dimension of $\mathfrak{kill}(U_i)$ stabilizes for i large enough. The resulting Lie algebra will be denoted by $\mathfrak{kill}(x)$.

Starting from a point $x \in M$, we can consider the set of all points $y \in M$ that can be reached from x by flowing along successive local Killing fields. This set is called the $\mathrm{Kill}^{\mathrm{loc}}$-orbit of x and is denoted as $\mathcal{O}_x^{\mathrm{loc}}$. Let us recall that for "generic" rigid structures, there are no local Killing fields at all, and the $\mathrm{Kill}^{\mathrm{loc}}$-orbits are reduced to points. The opposite situation is that of connected locally homogeneous structures, for which $\mathcal{O}_x^{\mathrm{loc}} = M$. An interesting weaker notion is that of quasihomogeneous structure.

Definition 1.3. — A geometric structure is called quasihomogeneous when the union of open $\mathrm{Kill}^{\mathrm{loc}}$-orbits is dense.

Gromov's theorem 1.1 says that a rigid geometric structure of algebraic type, with a topologically transitive automophism group, is quasihomogeneous. It is thus a question of general interest to understand when a quasihomogeneous structure is actually locally homogeneous.

It seems that there is no universal answer to this problem. For instance, A. Guillot and S. Dumitrescu exhibited in [7] quasihomogeneous affine connections on surfaces which are not homogeneous, *even in the real analytic category.*

On the contrary, S. Dumitrescu and K. Melnick recently showed in [8] that any real analytic Lorentz metric on a 3-manifold which is quasihomogeneous must be locally homogeneous. The analyticity assumption is crucial in their proof, and it is unknown whether the results of [8] still hold in the smooth category.

Actually, Theorem A will follow from a partial generalization of [8] to smooth manifolds. We will indeed show the following local result:

Theorem B. — *Let (M^3, g) be a smooth 3-dimensional Lorentz manifold (not necessarily closed). Assume that on a dense open subset, the Lie algebra of local Killing fields is at least 4-dimensional. Then (M^3, g) is locally homogeneous.*

It is not difficult to see that the hypothesis on the dimension of the local Killing algebras does imply quasihomogeneity of the metric (see Fact 4.1). For a quasihomogeneous Lorentz 3-manifold, the possible dimensions of the local Killing algebras $\mathfrak{kill}(x)$ are 3, 4 or 6 (in the smooth case, this dimension may vary with the point x). Hence Theorem B deals with quasihomogeneous structures without open $\mathrm{Kill}^{\mathrm{loc}}$-orbits having a 3-dimensional local Killing algebra.

Even if the conclusions of Theorem B are the same as for analytic metrics, the result cannot be obtained by adapting the methods of [8]. In fact, we would like to point out that those regularity issues constitute a great part of the subtleties in this type of problem. To emphasize this aspect, we observe that the proof of Theorem B works for metrics of class C^9 (this regularity is required since we will need several covariant derivatives of the curvature tensor). This regularity is probably not optimal, but let us stress that the conclusions change dramatically if we work with metrics that have too low regularity.

Theorem C. — *There exist 3-dimensional Lorentz manifolds (M^3, g) such that g is C^1 and quasihomogeneous and satisfies hypotheses of Theorem B, but is not locally homogeneous. Moreover, one can build compact examples in regularity C^0.*

The conclusions of Theorem C will be made more precise in Section 6 (see Theorems 6.1 and 6.2).

1.3. Organization of the paper. — A key ingredient in Gromov's theory of rigid transformation groups is a theorem regarding integration of finite-order Killing fields. We will make systematic use of this result in all our proofs, so that Section 2 will be devoted to a presentation of this theorem, in the convenient framework of Cartan geometries (which includes, of course, the case of Lorentz metrics). Next, we will use this integration result in Section 3 to prove two general criteria allowing us to show that some quasihomogeneous pseudo-Riemannian structures are actually locally homogeneous. Section 4 begins with a general study of local Lorentz actions of 4-dimensional Lie algebras on 3-manifolds. This study, together with the criteria established in Section 3, leads to a proof of Theorem B. In Section 5, we explain how Theorem A can be deduced from Theorem B. The upshot is to show that when the isometry group of a pseudo-Riemannian manifold is topologically transitive, then numerous local Killing fields must appear (even if the isometry group is discrete, for instance). Finally, Section 6 will be devoted to the construction of examples of Theorem C.

2. Integration of finite-order Killing fields

The main tool for understanding the $\mathrm{Kill}^{\mathrm{loc}}$-orbits of a rigid geometric structure is a theorem about integration of finite-order Killing fields proved in [13][Section 1.6]. The results of [13] generalize former integrability theorems proved by K. Nomizu in [17] and I. Singer in [20]. Here we won't follow the