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A NOTE ON CRYSTALLINE LIFTINGS IN THE $\mathbb{Q}_p$ CASE

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ABSTRACT. — Let $p > 2$ be a prime. Let ρ be a crystalline representation of $G_{\mathbb{Q}_p}$ with distinct Hodge-Tate weights in $[0, p]$, such that its reduction $\overline{\rho}$ is upper triangular. Under certain conditions, we prove that $\overline{\rho}$ has an upper triangular crystalline lift ρ' such that $\mathrm{HT}(\rho') = \mathrm{HT}(\rho)$. The method is based on the author's previous work, combined with an inspiration from the work of Breuil-Herzig.

RÉSUMÉ (*Note sur les élévations cristallines dans le cas $\mathbb{Q}_p$*). — Soit $p > 2$ un premier. Soit ρ une représentation cristalline de $G_{\mathbb{Q}_p}$ avec des poids distincts de Hodge-Tate dans $[0, p]$, de telle sorte que sa réduction $\overline{\rho}$ soit triangulaire supérieure. Dans certaines conditions, nous prouvons que $\overline{\rho}$ a une élévation cristalline triangulaire supérieure ρ' telle que $\mathrm{HT}(\rho') = \mathrm{HT}(\rho)$. La méthode est basée sur le travail antérieur de l'auteur, combiné avec une inspiration de l'oeuvre de Breuil-Herzig.

1. Introduction

1.1. Overview. — Given (a lattice in) a crystalline representation, it is natural to study its reduction. Conversely, given a representation over an $\overline{\mathbb{F}}_p$-vector space, it is natural to consider its crystalline lifts. We are particularly interested with crystalline representations, because they will have applications to

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weight part of Serre's conjectures (see e.g., [6, 7, 3]). In general, both these questions are notoriously difficult. For example, given an $\overline{\mathbb{F}}_p$-representation, we do not even know if it has any crystalline lift. However, for applications to weight part of Serre's conjectures, we can *assume* at the beginning that certain $\overline{\mathbb{F}}_p$-representation already have at least one crystalline lift; the key point then is to show that it has some other *nicer* crystalline lift. And this is what we do in this paper.

To state our main result, we introduce some notations first. Let $G_{\mathbb{Q}_p} := \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ be the Galois group of $\mathbb{Q}_p$. Let $E/\mathbb{Q}_p$ be a finite extension, $\mathcal{O}_E$ the ring of integers, ω_E a fixed uniformizer, and $k_E = \mathcal{O}_E/\omega_E\mathcal{O}_E$ the residue field. We will use the following notations often, (CRYS):

- Let $p > 2$ be an odd prime. Let V be a crystalline representation of $G_{\mathbb{Q}_p}$ of E-dimension d, such that the Hodge-Tate weights $\mathrm{HT}(V) = \{0 = r_1 < \ldots < r_d \leq p\}$.
- Let $\rho = T$ be a $G_{\mathbb{Q}_p}$-stable $\mathcal{O}_E$-lattice in V, and $\hat{\mathfrak{M}} \in \mathrm{Mod}_{\mathfrak{S}_{\mathcal{O}_E}}^{\varphi,\hat{G}}$ the $(\varphi,\hat{G})$-module (with $\mathcal{O}_E$-coefficient) attached to T. Let $\overline{\rho} := T/\omega_E T$ be the reduction. Let $\overline{\hat{\mathfrak{M}}}$ be the reduction of $\hat{\mathfrak{M}}$, and $\overline{\mathfrak{M}}$ the reduction of $\mathfrak{M}$.

1.1.1. Theorem. — *With notations in (*CRYS*). Suppose that $\overline{\rho}$ is upper triangular, i.e., $\overline{\rho}$ is a successive extension of d characters: $\overline{\chi}_1, \ldots, \overline{\chi}_d$. Suppose $\overline{\chi}_i\overline{\chi}_j^{-1} \neq \overline{\varepsilon}_p, \forall i \neq j$, where $\overline{\varepsilon}_p$ is the reduction of the cyclotomic character. Then there exists an upper triangular crystalline representation ρ' such that $\overline{\rho}' \cong \overline{\rho}$, and* $\mathrm{HT}(\rho') = \mathrm{HT}(\rho)$ *as sets.*

Theorem 1.1.1 strengthens [3, Cor. 0.2(1)] in the $\mathbb{Q}_p$-case, and of course have direct application to weight part of Serre's conjectures as in *loc. cit.*. In our Theorem 1.1.1,

- we do not require the Condition (C-1) of [3, §3], and
- we only require a weaker version of Condition (C-2A) of [3, §6].
- Note that Condition (C-2B) of [3, §6] in general will never be satisfied in our current paper.

Let us also remark that Condition (C-1) seems to be the most difficult condition to remove in [3].

The proof of our theorem still uses results in [3] to study the possible shape of upper triangular reductions of crystalline representations. The difference in the current paper is a different crystalline lifting technique, which is inspired by some group theory developed in [1]. Roughly speaking, we can use the group theory to conjugate our upper triangular $\overline{\rho}$ to another upper triangular form, which can be lifted to an *ordinary* (in particular, upper triangular) crystalline representation via the result of [5]. The lifting process via *loc. cit.* is in some sense easier than those used in [3] (which is generalization of methods in [6, 7]). However, we can only apply this technique in the $\mathbb{Q}_p$-case, because it seems that we cannot apply the group theory in [1] to deal with general $K/\mathbb{Q}_p$ case for our problem. Let us remark that our current paper shows a much refined structure for upper triangular reductions of crystalline representations. It is also worth pointing out that our result gives a very *natural* example (see (4.1.2)) for some of the group theories in [1].

The paper is organized as follows. In Section 2, we review the theory of Kisin modules and $(\varphi, \hat{G})$-modules with $\mathcal{O}_E$-coefficients. In Section 3, we review the group theory in [1]. In Section 4, we study the shape of upper triangular torsion $(\varphi, \hat{G})$-modules, using results in [3], as well as techniques inspired by the group theory in Section 3. Finally in Section 5, we prove our crystalline lifting theorem.

1.2. Notations. — The notations in the following are taken directly from [3]. In particular, they are valid for any finite extension $K/\mathbb{Q}_p$ (and we use K_0 to denote the maximal unramified sub-extension of K, and k the residue field of K). See *loc. cit.* for any unfamiliar terms and more details.

In this paper, we sometimes use boldface letters (e.g., $\boldsymbol{e}$) to mean a sequence of objects (e.g., $\boldsymbol{e} = (e_1, \ldots, e_d)$ a basis of some module). We use Mat(?) to mean the set of matrices with elements in ?. We use notations like $[u^{r_1}, \ldots, u^{r_d}]$ to mean a diagonal matrix with the diagonal elements in the bracket. We use Id to mean the identity matrix. For a matrix A, we use diag$A$ to mean the diagonal matrix formed by the diagonal of A.

In this paper, *upper triangular* always means successive extension of rank-1 objects. We use notations like $\mathcal{E}(m_d, \ldots, m_1)$ (note the order of objects) to mean the set of all upper triangular extensions of rank-1 objects in certain categories. That is, m is in $\mathcal{E}(m_d, \ldots, m_1)$ if there is an increasing filtration $0 = \mathrm{Fil}^0 m \subset \mathrm{Fil}^1 m \subset \ldots \subset \mathrm{Fil}^d m = m$ such that $\mathrm{Fil}^i m / \mathrm{Fil}^{i-1} m = m_i, \forall 1 \leq i \leq d$.

We normalize the Hodge-Tate weights so that $\mathrm{HT}_\kappa(\varepsilon_p) = 1$ for any $\kappa : K \to \overline{\mathbb{Q}_p}$, where ε_p is the p-adic cyclotomic character.

We fix a system of elements $\{\pi_n\}_{n=0}^\infty$ in $\overline{K}$, where $\pi_0 = \pi$ is a uniformizer of K, and $\pi_{n+1}^p = \pi_n, \forall n$. Let $K_n = K(\pi_n), K_\infty = \bigcup_{n=0}^\infty K(\pi_n)$, and $G_\infty := \mathrm{Gal}(\overline{K}/K_\infty)$. We fix a system of elements $\{\mu_{p^n}\}_{n=0}^\infty$ in $\overline{K}$, where $\mu_1 = 1$, μ_p is a primitive p-th root of unity, and $\mu_{p^{n+1}}^p = \mu_{p^n}, \forall n$. Let $K_{p^\infty} = \bigcup_{n=0}^\infty K(\mu_{p^n})$, and $\hat{K} = K_{\infty, p^\infty} = \bigcup_{n=0}^\infty K(\pi_n, \mu_{p^n})$. Note that $\hat{K}$ is the Galois closure of K_∞, and let $\hat{G} = \mathrm{Gal}(\hat{K}/K)$, $H_K = \mathrm{Gal}(\hat{K}/K_\infty)$, and $G_{p^\infty} = \mathrm{Gal}(\hat{K}/K_{p^\infty})$. When $p > 2$, then $\hat{G} \simeq G_{p^\infty} \rtimes H_K$ and $G_{p^\infty} \simeq \mathbb{Z}_p(1)$, and so we can (and do) fix a topological generator τ of G_{p^∞}. And we can furthermore assume that $\mu_{p^n} = \frac{\tau(\pi_n)}{\pi_n}$ for all n.

Let $C = \hat{\overline{K}}$ be the completion of $\overline{K}$, with ring of integers $\mathcal{O}_C$. Let $R := \varprojlim \mathcal{O}_C/p$ where the transition maps are p-th power map. R is a valuation ring with residue field $\bar{k}$ ($\bar{k}$ is the residue field of C). R is a perfect ring of characteristic p. Let $W(R)$ be the ring of Witt vectors. Let $\underline{\epsilon} := (\mu_{p^n})_{n=0}^{\infty} \in R$, $\underline{\pi} = (\pi_n)_{n=0}^{\infty} \in R$, and let $[\underline{\epsilon}], [\underline{\pi}]$ be their Teichmüller representatives respectively in $W(R)$. We normalize the valuation on R so that $v_R(\underline{\pi}) = \frac{1}{e}$, where e is the ramification index of $K/\mathbb{Q}_p$.

There is a map $\theta : W(R) \to \mathcal{O}_C$ which is the unique universal lift of the map $R \to \mathcal{O}_C/p$ (projection of R onto the its first factor), and $\operatorname{Ker}\theta$ is a principle ideal generated by $\xi = [\overline{\omega}] + p$, where $\overline{\omega} \in R$ with $\omega^{(0)} = -p$, and $[\overline{\omega}] \in W(R)$ its Teichmüller representative. Let $B_{\mathrm{dR}}^+ := \varprojlim_n W(R)[\frac{1}{p}]/(\xi)^n$, and $B_{\mathrm{dR}} := B_{\mathrm{dR}}^+[\frac{1}{\xi}]$. Let $t := \log([\underline{\epsilon}])$, which is an element in B_{dR}^+. Let A_{cris} denote the p-adic completion of the divided power envelope of $W(R)$ with respect to $\operatorname{Ker}(\theta)$. Let $B_{\mathrm{cris}}^+ = A_{\mathrm{cris}}[1/p]$ and $B_{\mathrm{cris}} := B_{\mathrm{cris}}^+[\frac{1}{t}]$. The projection from R to $\bar{k}$ induces a projection $\nu : W(R) \to W(\bar{k})$, since $\nu(\operatorname{Ker}\theta) = pW(\bar{k})$, the projection extends to $\nu : A_{\mathrm{cris}} \to W(\bar{k})$, and also $\nu : B_{\mathrm{cris}}^+ \to W(\bar{k})[\frac{1}{p}]$. Write $I_+ B_{\mathrm{cris}}^+ := \operatorname{Ker}(\nu : B_{\mathrm{cris}}^+ \to W(\bar{k})[\frac{1}{p}])$, and for any subring $A \subseteq B_{\mathrm{cris}}^+$, write $I_+ A = A \cap \operatorname{Ker}(\nu)$.

Let $\mathfrak{S} := W(k)[\![u]\!]$, $E(u) \in W(k)[u]$ the minimal polynomial of π over $W(k)$, and S the p-adic completion of the PD-envelope of $\mathfrak{S}$ with respect to the ideal $(E(u))$. We can embed the $W(k)$-algebra $W(k)[u]$ into $W(R)$ by mapping u to $[\underline{\pi}]$. The embedding extends to the embeddings $\mathfrak{S} \hookrightarrow S \hookrightarrow A_{\mathrm{cris}}$.

2. Kisin modules and $(\varphi, \hat{G})$-modules

In this section, we briefly review some facts in the theory of Kisin modules and $(\varphi, \hat{G})$-modules with $\mathcal{O}_E$-coefficients. The materials in this section are based on works of [8, 10, 2, 6, 9] etc.. But here we only cite them in the form as in [3, §1], where the readers can find more detailed attributions.

2.1. Kisin modules and $(\varphi, \hat{G})$-modules with coefficients. — In this subsection, all the definitions and results are valid for any finite extension $K/\mathbb{Q}_p$.

Recall that $\mathfrak{S} = W(k)[\![u]\!]$ with the Frobenius endomorphism $\varphi_{\mathfrak{S}} : \mathfrak{S} \to \mathfrak{S}$ which acts on $W(k)$ via arithmetic Frobenius and sends u to u^p. Denote $\mathfrak{S}_{\mathcal{O}_E} := \mathfrak{S} \otimes_{\mathbb{Z}_p} \mathcal{O}_E$ and $\mathfrak{S}_{k_E} := \mathfrak{S} \otimes_{\mathbb{Z}_p} k_E = k[\![u]\!] \otimes_{\mathbb{F}_p} k_E$. We can extend $\varphi_{\mathfrak{S}}$ to $\mathfrak{S}_{\mathcal{O}_E}$ (resp. $\mathfrak{S}_{k_E}$) by acting on $\mathcal{O}_E$ (resp. k_E) trivially. Let r be any nonnegative integer.

- Let $'\mathrm{Mod}_{\mathfrak{S}_{\mathcal{O}_E}}^{\varphi}$ (called the category of Kisin modules of height r with $\mathcal{O}_E$-coefficients) be the category whose objects are $\mathfrak{S}_{\mathcal{O}_E}$-modules $\mathfrak{M}$, equipped with $\varphi : \mathfrak{M} \to \mathfrak{M}$ which is a $\varphi_{\mathfrak{S}_{\mathcal{O}_E}}$-semi-linear morphism such that the span of $\operatorname{Im}(\varphi)$ contains $E(u)^r \mathfrak{M}$. The morphisms in the category are $\mathfrak{S}_{\mathcal{O}_E}$-linear maps that commute with φ.

- Let $\mathrm{Mod}^{\varphi}_{\mathfrak{S}_{\mathcal{O}_E}}$ be the full subcategory of $'\mathrm{Mod}^{\varphi}_{\mathfrak{S}_{\mathcal{O}_E}}$ with $\mathfrak{M} \simeq \oplus_{i\in I}\mathfrak{S}_{\mathcal{O}_E}$ where I is a finite set. Let $\mathrm{Mod}^{\varphi}_{\mathfrak{S}_{k_E}}$ be the full subcategory of $'\mathrm{Mod}^{\varphi}_{\mathfrak{S}_{\mathcal{O}_E}}$ with $\mathfrak{M} \simeq \oplus_{i\in I}\mathfrak{S}_{k_E}$ where I is a finite set.

For any integer $n \geq 0$, write $n = (p-1)q(n) + r(n)$ with $q(n)$ and $r(n)$ the quotient and residue of n divided by $p-1$. Let $t^{\{n\}} = (p^{q(n)} \cdot q(n)!)^{-1} \cdot t^n$, we have $t^{\{n\}} \in A_{\mathrm{cris}}$.

We define a subring of B^+_{cris}, $\mathcal{R}_{K_0} := \left\{\sum_{i=0}^{\infty} f_i t^{\{i\}}, f_i \in S_{K_0}, f_i \to 0 \text{ as } i \to \infty\right\}$. Define $\hat{\mathcal{R}} := \mathcal{R}_{K_0} \cap W(R)$. Then $\hat{\mathcal{R}}$ is a φ-stable subring of $W(R)$, which is also G_K-stable, and the G_K-action factors through $\hat{G}$. Denote $\hat{\mathcal{R}}_{\mathcal{O}_E} := \hat{\mathcal{R}} \otimes_{\mathbb{Z}_p} \mathcal{O}_E$, $W(R)_{\mathcal{O}_E} := W(R) \otimes_{\mathbb{Z}_p} \mathcal{O}_E$, and extend the G_K-action and φ-action on them by acting on $\mathcal{O}_E$ trivially. Note that $\mathfrak{S}_{\mathcal{O}_E} \subset \hat{R}_{\mathcal{O}_E}$, and let $\varphi : \mathfrak{S}_{\mathcal{O}_E} \to \hat{R}_{\mathcal{O}_E}$ be the composite of $\varphi_{\mathfrak{S}_{\mathcal{O}_E}} : \mathfrak{S}_{\mathcal{O}_E} \to \mathfrak{S}_{\mathcal{O}_E}$ and the embedding $\mathfrak{S}_{\mathcal{O}_E} \to \hat{R}_{\mathcal{O}_E}$.

2.1.1. Definition. — Let $'\mathrm{Mod}^{\varphi,\hat{G}}_{\mathfrak{S}_{\mathcal{O}_E}}$ be the category (called the category of $(\varphi,\hat{G})$-modules of height r with $\mathcal{O}_E$-coefficients) consisting of triples $(\mathfrak{M}, \varphi_{\mathfrak{M}}, \hat{G})$ where,

1. $(\mathfrak{M}, \varphi_{\mathfrak{M}}) \in {'\mathrm{Mod}^{\varphi}_{\mathfrak{S}_{\mathcal{O}_E}}}$ is a Kisin module of height r;
2. $\hat{G}$ is a $\hat{\mathcal{R}}_{\mathcal{O}_E}$-semi-linear $\hat{G}$-action on $\hat{\mathfrak{M}} := \hat{\mathcal{R}}_{\mathcal{O}_E} \otimes_{\varphi,\mathfrak{S}_{\mathcal{O}_E}} \mathfrak{M}$;
3. $\hat{G}$ commutes with $\varphi_{\hat{\mathfrak{M}}} := \varphi_{\hat{\mathcal{R}}_{\mathcal{O}_E}} \otimes \varphi_{\mathfrak{M}}$;
4. Regarding $\mathfrak{M}$ as a $\varphi(\mathfrak{S}_{\mathcal{O}_E})$-submodule of $\hat{\mathfrak{M}}$, then $\mathfrak{M} \subseteq \hat{\mathfrak{M}}^{H_K}$;
5. $\hat{G}$ acts on the $\hat{\mathfrak{M}}/(I_+\hat{R})\hat{\mathfrak{M}}$ trivially.

A morphism between two $(\varphi,\hat{G})$-modules is a morphism in $\mathrm{Mod}^{\varphi}_{\mathfrak{S}_{\mathcal{O}_E}}$ which commutes with $\hat{G}$-actions.

We denote $\mathrm{Mod}^{\varphi,\hat{G}}_{\mathfrak{S}_{\mathcal{O}_E}}$ to be the full subcategory of $'\mathrm{Mod}^{\varphi,\hat{G}}_{\mathfrak{S}_{\mathcal{O}_E}}$ where $\mathfrak{M} \in \mathrm{Mod}^{\varphi}_{\mathfrak{S}_{\mathcal{O}_E}}$; and we denote $\mathrm{Mod}^{\varphi,\hat{G}}_{\mathfrak{S}_{k_E}}$ for the full subcategory of $'\mathrm{Mod}^{\varphi,\hat{G}}_{\mathfrak{S}_{\mathcal{O}_E}}$ where $\mathfrak{M} \in \mathrm{Mod}^{\varphi}_{\mathfrak{S}_{k_E}}$.

We can associate representations to $(\varphi,\hat{G})$-modules.

2.1.2. Theorem ([3, Thm. 1.2, Thm. 1.4]). — 1. *Suppose* $\hat{\mathfrak{M}} \in \mathrm{Mod}^{\varphi,\hat{G}}_{\mathfrak{S}_{\mathcal{O}_E}}$ *where* $\mathfrak{M}$ *is of* $\mathfrak{S}_{\mathcal{O}_E}$*-rank* d*, then*

$$\hat{T}(\hat{\mathfrak{M}}) := \mathrm{Hom}_{\hat{\mathcal{R}},\varphi}(\hat{\mathfrak{M}}, W(R))$$

is a finite free $\mathcal{O}_E$*-representation of* G_K *of rank* d*.*

2. *Suppose* $\hat{\mathfrak{M}} \in \mathrm{Mod}^{\varphi,\hat{G}}_{\mathfrak{S}_{k_E}}$ *where* $\mathfrak{M}$ *is of* $\mathfrak{S}_{k_E}$*-rank* d*, then*

$$\hat{T}(\hat{\mathfrak{M}}) := \mathrm{Hom}_{\hat{\mathcal{R}},\varphi}(\hat{\mathfrak{M}}, W(R) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p)$$

is a finite free k_E*-representation of* G_K *of dimension* d*.*