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FINITENESS OF TOTALLY GEODESIC EXCEPTIONAL DIVISORS IN HERMITIAN LOCALLY SYMMETRIC SPACES

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Abstract. — We prove that on a smooth complex surface which is a compact quotient of the bidisc or of the 2-ball, there is at most a finite number of totally geodesic curves with negative self-intersection. More generally, there are only finitely many exceptional totally geodesic divisors in a compact Hermitian locally symmetric space of noncompact type of dimension at least 2. This is deduced from a convergence result for currents of integration along totally geodesic subvarieties in compact Hermitian locally symmetric spaces, which itself follows from an equidistribution theorem for totally geodesic submanifolds in a locally symmetric space of finite volume.

Résumé (*Finitude du nombre de diviseurs totalement géodésiques exceptionnels dans les variétés localement symétriques hermitiennes*). — Nous prouvons que sur une surface complexe lisse qui est un quotient compact du bidisque ou de la boule de dimension 2, il n'y a qu'un nombre fini de courbes totalement géodésiques d'auto-intersection strictement négative. Plus généralement, il n'y a qu'un nombre fini de diviseurs totalement géodésiques exceptionnels dans une variété localement symétrique (de type non compact) hermitienne compacte de dimension au moins 2. Ces énoncés

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sont déduits d'un théorème de convergence de courants d'intégration le long de sous-variétés totalement géodésiques dans les variétés localement symétriques hermitiennes compactes, lui-même obtenu à partir d'un résultat d'équidistribution des sous-variétés totalement géodésiques dans les variétés localement symétriques de volume fini.

1. Introduction

Our motivation for writing this note comes from a question about totally geodesic curves in compact quotients of the 2-ball related to the so-called Bounded Negativity Conjecture. This conjecture states that if X is a smooth complex projective surface, there exists a number $b(X) \geq 0$ such that any negative curve on X has self-intersection at least $-b(X)$. On a Shimura surface X, i.e. an arithmetic compact quotient of the bidisc or of the 2-ball, one can ask whether such a conjecture holds for Shimura (totally geodesic) curves. In [2], using an inequality of Miyaoka [15], it was proved that on a quaternionic Hilbert modular surface, that is, a compact quotient of the bidisc, there are only a finite number of negative Shimura curves. The same question for Picard modular surfaces, i.e. quotients of the 2-ball, was open as we learned from discussions with participants of the MFO mini-workshops "Kähler Groups" (`http://www.mfo.de/occasion/1409a/www_view`) and "Negative Curves on Algebraic Surfaces" (`http://www.mfo.de/occasion/1409b/www_view`). See the report [9] and [2, Remarks 3.3 & 3.7]. There was a general feeling that this should follow from an equidistribution result about totally geodesic submanifolds in locally symmetric manifolds. Using such a result, we prove that this is indeed true (we include the already known case of the bidisc since this is also implied by the same method):

Theorem 1.1. — *Let X be a closed complex surface whose universal cover is biholomorphic to either the 2-ball or the bidisc. Then X only supports a finite number of totally geodesic curves with negative self-intersection.*

More generally, let X be a closed Hermitian locally symmetric space of non-compact type of complex dimension $n \geq 2$. Then X only supports a finite number of exceptional totally geodesic divisors.

It is known that the irreducible Hermitian symmetric spaces of noncompact type admitting totally geodesic divisors are those associated with the Lie groups $\mathrm{SU}(n,1)$, $n \geq 1$, and $\mathrm{SO}_0(p,2)$, $p \geq 3$, and then that the divisors are associated with the subgroups $\mathrm{SU}(n-1,1)$ and $\mathrm{SO}_0(p-1,2)$ respectively, see [18, 4]. Note, however, that Theorem 1.1 also applies in the case of reducible symmetric spaces.

The first assertion of this result has been obtained independently and at the same time by M. Möller and D. Toledo [16], who also participated in the aforementioned workshops. We refer to their paper for background on Shimura surfaces and Shimura curves, and in particular for a discussion of the arithmetic quotients of the 2-ball and of the bidisc, which admit infinite families of pairwise distinct totally geodesic curves. Their proof is based on an equidistribution theorem for curves in 2-dimensional Hermitian locally symmetric spaces [16, § 2]. Here we have chosen to present a more general result, see Theorem 1.2 below, in the hope that it can be useful in a wider setting (and indeed it implies the second assertion of the theorem).

Henceforth we will be interested in closed totally geodesic (possibly singular) submanifolds in non-positively curved locally symmetric manifolds of finite volume.

Let $\mathcal{X}$ be a symmetric space of noncompact type, $G = \mathrm{Isom}_0(\mathcal{X})$ the connected component of the isometry group of $\mathcal{X}$, Γ a torsion-free lattice of G and X the quotient locally symmetric manifold $\Gamma\backslash\mathcal{X}$.

Complete connected totally geodesic (smooth) submanifolds of $\mathcal{X}$ are naturally symmetric spaces themselves, and we will call such a subset $\mathcal{Y}$ a symmetric subspace of noncompact type of $\mathcal{X}$ if as a symmetric space it is of noncompact type, i.e. it has no Euclidean factor. Up to the action of G, there is only a finite number of symmetric subspaces of noncompact type in $\mathcal{X}$, see Fact 2.4. The orbit of $\mathcal{Y}$ under G will be called the *kind* of $\mathcal{Y}$.

A subset Y of X will be called a *closed totally geodesic submanifold of noncompact type* of X if it is of the form $\Gamma\backslash\Gamma\mathcal{Y}$, where $\mathcal{Y}$ is a symmetric subspace of noncompact type of $\mathcal{X}$ such that if $S_{\mathcal{Y}} < G$ is the stabilizer of $\mathcal{Y}$ in G, $\Gamma\cap S_{\mathcal{Y}}$ is a lattice in $S_{\mathcal{Y}}$. The *kind* of $Y = \Gamma\backslash\Gamma\mathcal{Y}$ is by definition the kind of $\mathcal{Y}$.

It will simplify the exposition to consider only symmetric subspaces of $\mathcal{X}$ passing through a fixed point $o \in \mathcal{X}$. Therefore, we define equivalently a closed totally geodesic submanifold of noncompact type Y of X to be a subset of the form $\Gamma\backslash\Gamma g\mathcal{Y}$, where $\mathcal{Y} \subset \mathcal{X}$ is a symmetric subspace of noncompact type passing through $o \in \mathcal{X}$, and $g \in G$ is such that if $S_{\mathcal{Y}} < G$ is the stabilizer of $\mathcal{Y}$ in G, $\Gamma \cap gS_{\mathcal{Y}}g^{-1}$ is a lattice in $gS_{\mathcal{Y}}g^{-1}$.

Such a Y is indeed a closed totally geodesic submanifold of X, which might be singular, and it supports a natural probability measure μ_Y which can be defined as follows. By assumption, the (right) $S_{\mathcal{Y}}$-orbit $\Gamma\backslash\Gamma gS_{\mathcal{Y}} \subset \Gamma\backslash G$ is closed and supports a unique $S_{\mathcal{Y}}$-invariant probability measure ([19, Chap. 1]). We will denote by μ_Y the probability measure on X whose support is Y and which is defined as the push forward of the previous measure by the projection $\pi : \Gamma\backslash G \longrightarrow X = \Gamma\backslash G/K$, where K is the isotropy subgroup of G at o. In the special case when $S_{\mathcal{Y}} = G$, we obtain the natural probability measure μ_X on X.

We will say that a closed totally geodesic submanifold of noncompact type $Y = \Gamma\backslash\Gamma g\mathcal{Y}$ as mentioned above is a *local factor* if $\mathcal{Y} \subset \mathcal{X}$ is a *factor*, meaning that there exists a totally geodesic isometric embedding $f : \mathcal{Y} \times \mathbb{R} \longrightarrow \mathcal{X}$ such that $f(y, 0) = y$ for all $y \in \mathcal{Y}$.

We may now state the following.

THEOREM 1.2. — *Let $\mathcal{X}$ be a symmetric space of noncompact type, Γ a torsion-free lattice of the connected component G of its isometry group and X the quotient manifold $\Gamma\backslash\mathcal{X}$. Let $(Y_j)_{j\in\mathbb{N}}$ be a sequence of closed totally geodesic submanifolds of noncompact type of X. Assume that no subsequence of $(Y_j)_{j\in\mathbb{N}}$ is either composed of local factors or contained in a closed totally geodesic proper submanifold of X.*

Then the sequence of probability measures $(\mu_{Y_j})_{j\in\mathbb{N}}$ converges to the probability measure μ_X.

REMARK 1.3. — Although we have not been able to find its exact statement in the literature, this theorem is certainly known to experts in homogeneous dynamics and follows from several equidistribution results originating in the work of M. Ratner on unipotent flows, see in particular the work of A. Eskin, S. Mozes and N. Shah [17] and [11]. In the case of special subvarieties of Shimura varieties, a very similar result has been obtained by L. Clozel and E. Ullmo [8, 22]. From the perspective of geodesic flows (which is in a sense orthogonal to unipotent flows), Theorem 1.2 can probably also be deduced from A. Zeghib's article [24] (at least for ball quotients it can be).

The proof we give here is based on a result of Y. Benoist and J.-F. Quint [3], see Section 3.1. As we just said, anterior results certainly imply Theorem 1.2 and moreover the scope of [3] is far larger than the problem at hand, but to our mind, the way the result of [3] is formulated makes it easier to apply to our situation.

REMARK 1.4. — A symmetric subspace $\mathcal{Y} \subset \mathcal{X}$ of noncompact type is the orbit of a point in $\mathcal{X}$ under a connected semisimple subgroup without compact factors $H_{\mathcal{Y}}$ of $G = \mathrm{Isom}_0(\mathcal{X})$. The assumption that $\mathcal{Y}$ is not a factor means that the centralizer $Z_G(H_{\mathcal{Y}})$ of $H_{\mathcal{Y}}$ in G is compact. Another equivalent formulation is that $\mathcal{Y}$ is the only totally geodesic orbit of $H_{\mathcal{Y}}$ in $\mathcal{X}$. See Fact 2.2 for a proof.

This assumption seems quite strong, but the conclusion of Theorem 1.2 is false in general without it, as the following simple example shows. Let $X = \Sigma_1 \times \Sigma_2$ be the product of two Riemann surfaces of genus at least 2 and let $(z_j)_{j\in\mathbb{N}}$ be a sequence of distinct points in Σ_1 such that no subsequence is contained in a proper geodesic of Σ_1. Set $Y_j = \{z_j\} \times \Sigma_2$. Then for any subsequence of (z_j) converging to some $z \in \Sigma_1$, the corresponding subsequence of measures μ_{Y_j} converges to $\mu_{\{z\}\times\Sigma_2}$.

We observe that for rank 1 symmetric spaces, and in the case of uniform *irreducible* lattices of the bidisc, the assumption is automatically satisfied (see the proof of Theorem 1.1 in Section 3.4).

It would be interesting to know whether it is still needed if one assumes e.g. that $\mathcal{X}$ or Γ is irreducible.

In the case of Hermitian locally symmetric spaces, Theorem 1.2 gives a convergence result for currents of integration along closed complex totally geodesic subvarieties (suitably renormalized) from which Theorem 1.1 will follow. Recall that on a complex manifold X of dimension n, a current T of bidegree $(n-p, n-p)$ is said to be *(weakly) positive* if for any choice of smooth $(1,0)$-forms $\alpha_1, \ldots, \alpha_p$ on X, the distribution $T \wedge i\alpha_1 \wedge \bar{\alpha}_1 \wedge \cdots \wedge i\alpha_p \wedge \bar{\alpha}_p$ is a positive measure.

Corollary 1.5. — *Let X and $(Y_j)_{j\in\mathbb{N}}$ satisfy the assumptions of Theorem 1.2 and assume in addition that X is a compact Hermitian locally symmetric space of complex dimension n and that the Y_js are complex p-dimensional subvarieties of X of the same kind.*

Then there exists a closed positive $(n-p, n-p)$-form Ω on X (in the sense of currents), induced by a G-invariant $(n-p, n-p)$-form on $\mathcal{X} = G/K$, such that for any (p,p)-form η on X,

$$\lim_{j\to+\infty} \frac{1}{\operatorname{vol}(Y_j)} \int_{Y_j} \eta = \frac{1}{\operatorname{vol}(X)} \int_X \eta \wedge \Omega$$

Moreover, up to a positive constant, Ω depends only on the kind of the Y_js and if the Y_js are divisors, i.e. if $p = n-1$, then for any j, the $(1,1)$-form Ω restricted to Y_j does not vanish.

Since our initial interest was in 2-ball quotients, we underline that ball quotients X satisfying the assumptions of this corollary exist: the arithmetic manifolds whose fundamental groups are the so-called uniform lattices of type I in the automorphism group $\mathrm{PU}(n,1)$ of the n-ball are examples of manifolds supporting infinitely many complex totally geodesic subvarieties of dimension p for each $1 \leq p < n$, not all contained in a proper totally geodesic subvariety. Moreover, any complex p-dimensional totally geodesic subvariety of X is itself a quotient of the p-ball and, as already mentioned, is not a local factor in X (because the n-ball is a rank 1 symmetric space).

In the case of 2-ball quotients, the form Ω of Corollary 1.5 is proportional to the Kähler form induced by the unique (up to a positive constant) $\mathrm{SU}(2,1)$-invariant Kähler form on the ball $\mathbb{H}^2_{\mathbb{C}}$. In the case of quotients of the bidisc, and if ω denotes the unique (up to a positive constant) $\mathrm{SU}(1,1)$-invariant Kähler form on $\mathbb{H}^1_{\mathbb{C}}$, Ω is proportional to the Kähler form induced by the $\mathrm{SU}(1,1) \times$