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ABOUT THE BEHAVIOR OF REGULAR NAVIER-STOKES SOLUTIONS NEAR THE BLOW UP

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Abstract. — In this paper, we present some results about blow up of regular solutions to the homogeneous incompressible Navier-Stokes system, in the case of data in the Sobolev space $\dot{H}^s(\mathbb{R}^3)$, where $\frac{1}{2} < s < \frac{3}{2}\cdot$ Firstly, we will introduce the notion of minimal blow up Navier-Stokes solutions and show that the set of such solutions is not only nonempty but also compact in a certain sense. Secondly, we will state an uniform blow up rate for minimal Navier-Stokes solutions. The key tool is profile theory as established by P. Gérard [11].

1. Introduction

We consider the Navier-Stokes system for incompressible fluids evolving in the whole space $\mathbb{R}^3$. Denoting by u the velocity, a vector field in $\mathbb{R}^3$, by p in $\mathbb{R}$ the pressure function, the Cauchy problem for the homogeneous incompressible Navier-Stokes system is given by

$$(1) \qquad \begin{cases} \partial_t u + u \cdot \nabla u - \Delta u = -\nabla p \\ \operatorname{div} u = 0 \\ u_{|t=0} = u_0. \end{cases}$$

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Throughout this paper, we will adopt the useful notation $NS(u_0)$ to mean the maximal solution of the Navier-Stokes system, associated with the initial data u_0.

DEFINITION 1.1. — Let s in $\mathbb{R}$. The homogeneous Sobolev space $\dot{H}^s(\mathbb{R}^3)$ is the space of tempered distributions u over $\mathbb{R}^3$, the Fourier transform of which belongs to $L^1_{loc}(\mathbb{R}^3)$ and satisfies

$$\|u\|_{\dot{H}^s} \stackrel{\text{def}}{=} \left(\int_{\mathbb{R}^3} |\xi|^{2s} |\widehat{u}(\xi)|^2 d\xi\right)^{\frac{1}{2}} < \infty.$$

It is known that $\dot{H}^s(\mathbb{R}^3)$ is an Hilbert space if and only if $s < \frac{3}{2}$. We will denote by $(\cdot|\cdot)_{\dot{H}^s(\mathbb{R}^3)}$, the scalar product in $\dot{H}^s(\mathbb{R}^3)$. From now on, for the sake of simplicity, it will be an implicit understanding that all computations will be done in the whole space $\mathbb{R}^3$.

Before stating the results we prove in this paper, we recall two fundamental properties of the incompressible Navier-Stokes system. The first one is the conservation of the L^2 energy. Formally, let us take the L^2 scalar product with the velocity u in the equation. We get

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{L^2} + \|\nabla u(t)\|^2_{L^2} = -\int_{R^3} \big(u\cdot \nabla u(t)|u(t)\big)_{L^2} - \int_{R^3} \big(\nabla p(t)|u(u)\big)_{L^2}. \tag{2}$$

Thanks to the divergence free condition, obvious integration by parts implies that, for any vector field a

$$\big(u\cdot \nabla a|a\big)_{L^2} = 0 = \big(\nabla p|a\big)_{L^2}. \tag{3}$$

This gives

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{L^2} + \|\nabla u(t)\|^2_{L^2} = 0. \tag{4}$$

The second property of the system is the scaling invariance. Let us define the operator:

$$\begin{gathered}\forall \alpha \in \mathbb{R}^+,\ \forall \lambda \in \mathbb{R}^+_*, \forall x_0 \in \mathbb{R}^3,\ \ \Lambda^\alpha_{\lambda,x_0} u(t,x) \stackrel{\text{def}}{=} \frac{1}{\lambda^\alpha} u\Big(\frac{t}{\lambda^2}, \frac{x-x_0}{\lambda}\Big).\\ \text{If } \alpha = 1, \text{ we note } \Lambda^1_{\lambda,x_0} = \Lambda_{\lambda,x_0}.\end{gathered} \tag{5}$$

It is easy to see that if u is a smooth solution of Navier-Stokes system on $[0,T]\times\mathbb{R}^3$ with pressure p associated with the initial data u_0, then, for any positive λ, the vector field and the pressure

$$u_\lambda \stackrel{\text{def}}{=} \Lambda_{\lambda,x_0} u \text{ and } p_\lambda \stackrel{\text{def}}{=} \Lambda^2_{\lambda,x_0} p$$

is a solution of Navier-Stokes system on the interval $[0,\lambda^2 T]\times\mathbb{R}^3$, associated with the initial data

$$u_{0,\lambda} = \Lambda_{\lambda,x_0} u_0.$$

This leads to the definition of scaling invariant space, which is a key notion to investigate local and global well-posedness issues for Navier-Stokes system.

DEFINITION 1.2. — A Banach space X is said to be scaling invariant, if its norm is invariant under the scaling transformation defined by $u \mapsto u_\lambda$

$$\|u_\lambda\|_X = \|u\|_X$$

The first main result on incompressible Navier-Stokes system is due to J. Leray, who proved in [19] in 1934 that given an initial data in the energy space L^2, the associated NS-solutions, called weak solutions, exist globally in time. The key ingredient of the proof is the L^2-energy conservation (4). Moreover, such solutions are unique in 2-D; but the uniqueness in 3-D is still an open problem. One way to adress this question of unique solvability in 3-D is to demand smoother initial data. In this case, we definitely get a unique solution, but the other side of coin is that the problem is only locally well-posed (and becomes globally well-posed under a scaling invariant smallness assumption on the initial data). J. Leray stated such a theorem of existence of solutions, which he called semi-regular solutions.

THEOREM 1.1. — *Let an initial data u_0 be a divergence free vector field in L^2 such that ∇u_0 belongs to L^2. Then, there exists a positive time T, and a unique solution $NS(u_0)$ in $\mathcal{C}^0([0,T],\dot{H}^1) \ \cap \ L^2([0,T],\dot{H}^2)$.*

Moreover, a constant c_1 exists such that if $\|u_0\|_{L^2}\,\|\nabla u_0\|_{L^2} \leqslant c_1$, then T can be chosen equal to ∞.

The reader will have noticed that the quantity $\|u_0\|_{L^2}\,\|\nabla u_0\|_{L^2}$ is scaling invariant under the operator Λ_{λ,x_0}. Actually, that is the starting point of many frameworks concerning the global existence in time of solutions under a scaling invariant smallness assumption on the data. The celebrated first one was introduced in 1964, by H. Fujita and T. Kato. These authors stated a similar result as J. Leray, but they demanded less regularity on the data. Indeed, they proved that for any initial data in $\dot{H}^{\frac{1}{2}}$, there exists a positive time T and there exists a unique solution $NS(u_0)$ belonging to $\mathcal{C}^0([0,T],\dot{H}^{\frac{1}{2}}) \cap L^2([0,T],\dot{H}^{\frac{3}{2}})$. Moreover, if $\|u_0\|_{\dot{H}^{\frac{1}{2}}}$ is small enough, then the solution is global in time. This theorem can be proved by a fixed-point argument and the key ingredient of the proof is that the Sobolev space $\dot{H}^{\frac{1}{2}}$ is invariant under the operator Λ_{λ,x_0}. In other words, the Sobolev space $\dot{H}^{\frac{1}{2}}$ has exactly the same scaling as Navier-Stokes equation. We refer the reader to [1], [7] or [18] for more details of the proof. But in this paper, we work with initial data belonging to homogeneous Sobolev spaces, $\dot{H}^s$ with $\frac{1}{2} < s < \frac{3}{2}$, which means that we are above the natural scaling of the equation. The first thing to do is to provide an existence theorem of Navier-Stokes solutions with data in such Sobolev spaces $\dot{H}^s$. The Cauchy problem is known to be locally well-posed; it can be proved by a fixed-point procedure in an adequate function space (we refer the reader to the book [18], from page 146 to 148, of P-G. Lemarié-Rieusset).

We shall constantly be using the following simplified notations:

$$L^\infty_T(\dot H^s) \overset{\text{def}}{=} L^\infty([0,T],\dot H^s) \text{ and } L^2_T(\dot H^{s+1}) \overset{\text{def}}{=} L^2([0,T],\dot H^{s+1}).$$

Let us define the relevant function space we shall be working with in the sequel:

$$X^s_T \overset{\text{def}}{=} L^\infty_T(\dot H^s) \cap L^2_T(\dot H^{s+1}),\ \text{equipped with } \|u\|^2_{X^s_T} \overset{\text{def}}{=} \|u\|^2_{L^\infty_T(\dot H^s)} + \|u\|^2_{L^2_T(\dot H^{s+1})}.$$

THEOREM 1.2. — *Let u_0 be in $\dot H^s$, with $\frac12 < s < \frac32$. Then there exists a time T and there exists a unique solution $NS(u_0)$ such that $NS(u_0)$ belongs to $L^\infty_T(\dot H^s) \cap L^2_T(\dot H^{s+1})$.*

Moreover, let $T_*(u_0)$ be the maximal time of existence of such a solution. Then, there exists a positive constant c such that

$$T_*(u_0)\ \|u_0\|^{\sigma_s}_{\dot H^s} \geqslant c,\ \text{with}\ \sigma_s \overset{\text{def}}{=} \frac{1}{\frac12(s-\frac12)}. \tag{6}$$

REMARK 1.1. — As a by-product of the proof of Picard's Theorem, we get actually for free the following property: if the initial data is small enough (in the sense of there exists a positive constant c_0, such that $T\ \|u_0\|^{\sigma_s}_{\dot H^s} \leqslant c_0$), then a unique Navier-Stokes solution associated with it exists (locally in time, until the blow up time given by the relation (6)) and satisfies the following linear control

$$\forall\ 0 \leqslant T \leqslant \frac{c_0}{\|u_0\|^{\sigma_s}_{\dot H^s}},\ \|NS(u_0)(t,\cdot)\|_{X^s_T} \leqslant 2\,\|u_0\|_{\dot H^s}. \tag{7}$$

Formula (6) invites us to consider the lower boundary, denoted by $A^{\sigma_s}_s$, of the lifespan of such a solution

$$A^{\sigma_s}_s \overset{\text{def}}{=} \inf\left\{\ T_*(u_0)\|u_0\|^{\sigma_s}_{\dot H^s}\ \mid\ u_0 \in \dot H^s\ ;\ T_*(u_0) < \infty\right\}.$$

Obviously, $A^{\sigma_s}_s$ exists and is a positive real number and we always have the formula

$$T_*(u_0)\|u_0\|^{\sigma_s}_{\dot H^s} \geqslant A^{\sigma_s}_s. \tag{8}$$

Throughout this paper, we make the assumption of blow up, which is still an open problem. More precisely, we claim the following hypothesis.

Hypothesis $\mathcal{H}$: We consider s in $\left]\frac12,\frac32\right[$, such that a divergence-free vector field u_0 exists in $\dot H^s$ with a finite the lifespan $T_*(u_0)$.

DEFINITION 1.3 (Minimal blow up solution). — We say that $u = NS(u_0)$ is a minimal blow up solution if u_0 satisfies

$$T_*(u_0)\|u_0\|^{\sigma_s}_{\dot H^s} = A^{\sigma_s}_s.$$

In other terms $u = NS(u_0)$ is a minimal blow up solution if and only if $A_s^{\sigma_s}$ is reached.

Question: Under Hypothesis $\mathcal{H}$, do some minimal blow up solutions exist?

We will prove a stronger result: the set of initial data generating minimal blow up solutions with blow up time T_*, denoted by $\mathcal{M}_s(T_*)$, is not only a nonempty subset of $\dot{H}^s$ (which, in particular, gives the positive answer to the question) but also compact in a sense which is given in Theorem 1.3.

THEOREM 1.3. — *Assuming hypothesis $\mathcal{H}$, for any finite time T_*, the set $\mathcal{M}_s(T_*)$ is non empty and compact, up to translations. This means that for any sequence $(u_{0,n})_{n\in\mathbb{N}}$ of points in the set $\mathcal{M}_s(T_*)$, a sequence $(x_n)_{n\in\mathbb{N}}$ of points of $(\mathbb{R}^3)^{\mathbb{N}}$ and a function V in $\mathcal{M}_s(T_*)$ exist such that, up to an extraction*

$$\lim_{n\to+\infty} \|u_{0,n}(\cdot + x_n) - V\|_{\dot{H}^s} = 0.$$

The second result of this paper states that the blow up rate of a minimal blow up solution can be uniformly controlled since we get a priori bound of these minimal blow up solutions.

THEOREM 1.4 (Control of minimal blow up solutions). — *Assuming $\mathcal{H}$, there exists a nondecreasing function $F_s : [0, A_s^{\sigma_s}[\to \mathbb{R}^+$ with $\lim_{r\to A_s^{\sigma_s}} F_s(r) = +\infty$ such that for any divergence free vector field u_0 in $\dot{H}^s$, generating minimal blow up solution (it means $T_*(u_0)\|u_0\|_{\dot{H}^s}^{\sigma_s} = A_s^{\sigma_s}$), we have the following control on the minimal blow up solution $NS(u_0)$*

$$\forall T < T_*(u_0),\ \|NS(u_0)\|_{X_T^s} \leqslant \|u_0\|_{\dot{H}^s} F_s(T^{\frac{1}{\sigma_s}} \|u_0\|_{\dot{H}^s}).$$

REMARK 1.2. — Let us point out that the quantity $T^{\frac{1}{\sigma_s}} \|u_0\|_{\dot{H}^s}$ is scaling invariant; which is obviously necessary.

The two previous theorems are the analog of results, proved in the case of the Sobolev space $\dot{H}^{\frac{1}{2}}$. We shall not recall all the statements existing in the literature concerning the regularity of Navier-Stokes solutions in critical spaces, such as $\dot{H}^{\frac{1}{2}}$. We refer for instance the reader to [7] and to the article of C. Kenig et G. Koch [13], where the authors prove that NS-solutions which remain bounded in the space $\dot{H}^{\frac{1}{2}}$ do not become singular in finite time. Concerning Theorem 1.3, we were largely inspired by the article of W. Rusin and V. Šverák [23], in which the authors set up the key concept of minimal blow-up for data in Sobolev space $\dot{H}^{\frac{1}{2}}$. Firstly, they defined a critical radius $\rho_{\frac{1}{2}}$

$$\rho_{\frac{1}{2}} = \sup\big\{\rho > 0\,;\, \|u_0\|_{\dot{H}^{\frac{1}{2}}} < \rho \implies T_*(u_0) = +\infty\big\}.$$

Then, they introduced a subset $\mathcal{M}$ of $\dot{H}^{\frac{1}{2}}$, which describes the set of minimal-norm singularities (we speak about minimal norm in the sense of $\|u_0\|_{\dot{H}^{\frac{1}{2}}}$ is