Bull. Soc. Math. France
146 (2), 2018, p. 309-353

COMPACT DOMAINS WITH PRESCRIBED CONVEX BOUNDARY METRICS IN QUASI-FUCHSIAN MANIFOLDS

BY DMITRIY SLUTSKIY

ABSTRACT. — We show the existence of a convex compact domain in a quasi-Fuchsian manifold such that the induced metric on its boundary coincides with a prescribed surface metric of curvature $K \geq -1$ in the sense of A. D. Alexandrov.

This result extends the existence part of the classical result by Alexandrov and Pogorelov on the realization of a convex domain with a prescribed boundary metric in $\mathbb{H}^3$ in the case where $\mathbb{H}^3$ is replaced by a quasi-Fuchsian manifold and therefore the topology of a convex domain is not trivial.

RÉSUMÉ (*Domaines convexes compacts avec des métriques de bord prescrites dans les variétés quasi-fuchsiennes*). — Nous montrons l'existence d'un tel domaine compact convexe dans une variété quasi-fuchsienne que la métrique induite sur son bord coïncide avec une métrique prescrite de courbure $K \geq -1$ au sens de A. D. Alexandrov.

Ce résultat étend la partie d'existence d'un résultat classique par Alexandrov et Pogorelov sur la réalisation d'un domaine convexe avec une metrique de bord prescrite dans $\mathbb{H}^3$ dans le cas où $\mathbb{H}^3$ est remplacé par une variété quasi-fuchsienne et donc la topologie d'un domaine convexe n'est pas triviale.

Texte reçu le 1er février 2016, modifié le 13 juillet 2016, accepté le 6 octobre 2016.

DMITRIY SLUTSKIY, Université de Cergy-Pontoise, Laboratoire AGM, 2 av. Adolphe Chauvin, 95302 Cergy-Pontoise Cedex, France • *E-mail :* `Dmitriy.Slutskiy@u-cergy.fr`

Mathematical subject classification (2010). — 53C45, 20H10; 53C42, 30F40, 57M10, 57M40.

Key words and phrases. — quasi-Fuchsian manifold, convex compact domain, Alexandrov space, induced metric.

The author acknowledges support from U.S. National Science Foundation grants DMS 1107452, 1107263, 1107367 "RNMS: Geometric Structures and Representation Varieties" (the GEAR Network).

 0037-9484/2018/309/$ 5.00
 doi:10.24033/bsmf.2759

1. Introduction

The problem of existence and uniqueness of an isometric realization of a surface with a prescribed metric in a given ambient space is classical in the metric geometry. Initially stated in the Euclidean case, it can be posed for surfaces in other spaces, in particular, in hyperbolic 3-space $\mathbb{H}^3$.

One of the first fundamental results in this theory is due to A. D. Alexandrov. It concerns the realization of polyhedral surfaces in the spaces of constant curvature.

As in [22], we denote by $M^m(K)$ the m-dimensional complete simply connected space of constant sectional curvature K. So, $M^3(K)$ stands for spherical 3-space of curvature K in the case $K > 0$; $M^3(K)$ stands for hyperbolic 3-space of curvature K when $K < 0$; and in the case $K = 0$, $M^3(K)$ denotes Euclidean 3-space.

Then the result of A. D. Alexandrov reads as follows:

Theorem 1.1 ([3]). — *Let h be a metric of constant sectional curvature K with cone singularities on a sphere S^2 such that the total angle around every singular point of h does not exceed 2π. Then there exists a closed convex polyhedron in $M^3(K)$ equipped with the metric h which is unique up to the isometries of $M^3(K)$. Here we include the doubly covered convex polygons, which are planar in $M^3(K)$, in the set of convex polyhedra.*

Later, A. D. Alexandrov and A. V. Pogorelov proved the following statement in $\mathbb{H}^3$ [19]:

Theorem 1.2. — *Let h be a C^∞-regular metric of sectional curvature which is strictly greater than -1 on a sphere S^2. Then there exists an isometric immersion of the sphere (S^2, h) into hyperbolic 3-space $\mathbb{H}^3$ which is unique up to the isometries of $\mathbb{H}^3$. Moreover, this immersion bounds a convex domain in $\mathbb{H}^3$.*

Definition 1.1 ([15, p. 30], [17, p. 11]). — A discrete finitely generated subgroup $\Gamma_F \subset PSL_2(\mathbb{R})$ without torsion and such that the quotient $\mathbb{H}^2/\Gamma_F$ has a finite volume, is called a *Fuchsian group*.

Given a hyperbolic plane $\mathcal{P}$ in $\mathbb{H}^3$ and a Fuchsian group $\Gamma_{\mathcal{P}} \subset PSL_2(\mathbb{R})$ acting on $\mathcal{P}$, we can canonically extend the action of the group $\Gamma_{\mathcal{P}}$ on the whole space $\mathbb{H}^3$.

Here we recall another result on the above-mentioned problem considered for a special type of hyperbolic manifolds, namely, for Fuchsian manifolds, which is due to M. Gromov [12]:

Theorem 1.3. — *Let S be a compact surface of genus greater than or equal to 2, equipped with a C^∞-regular metric h of sectional curvature which is greater* *than* -1 *everywhere. Then there exists a Fuchsian group* Γ_F *acting on* $\mathbb{H}^3$*, such that the surface* (S,h) *is isometrically embedded in* $\mathbb{H}^3/\Gamma_F$*.*

REMARK 1.4. — The hyperbolic manifold $\mathbb{H}^3/\Gamma_F$ from the statement of Theorem 1.3 is called Fuchsian. Note also that the limit set $\Lambda(\Gamma_F) \subset \partial_\infty \mathbb{H}^3$ of a Fuchsian group Γ_F is a geodesic circle in projective space $\mathbb{CP}^1$ regarded as the boundary at infinity $\partial_\infty \mathbb{H}^3$ of the Poincaré ball model of hyperbolic 3-space $\mathbb{H}^3$.

In 2007 F. Fillastre [9] proved a polyhedral analog of Theorem 1.3, i.e., when h is a hyperbolic metric with cone singularities of angle less than 2π (the term "hyperbolic" means for us "of constant curvature equal to -1 everywhere").

DEFINITION 1.2 ([13]). — A compact hyperbolic manifold M is said to be *strictly convex* if any two points in M can be joined by a minimizing geodesic which lies inside the interior of M. This condition implies that the intrinsic curvature of ∂M is greater than -1 everywhere.

In 1992 F. Labourie [13] obtained the following result which can be considered as a generalization of Theorems 1.2 and 1.3:

THEOREM 1.5. — *Let* M *be a compact manifold with boundary (different from the solid torus) which admits a structure of a strictly convex hyperbolic manifold. Let* h *be a* C^∞*-regular metric on* ∂M *of sectional curvature which is strictly greater than* -1 *everywhere. Then there exists a convex hyperbolic metric* g *on* M *which induces* h *on* ∂M*:*

$$g|_{\partial M} = h.$$

Recall that the *limit set* $\Lambda(\Gamma_F) \subset \partial_\infty \mathbb{H}^3$ of a Fuchsian group Γ_F acting on $\mathbb{H}^3$ is the intersection of some hyperbolic plane with the boundary at infinity of the hyperbolic 3-space $\mathbb{H}^3$, i.e., a circle (in the Poincaré and Klein models of the hyperbolic 3-space).

Particular examples of the varieties considered in Theorem 1.5 are the quasi-Fuchsian manifolds.

DEFINITION 1.3 ([15, p. 120]). — A *quasi-Fuchsian manifold* is a quasiconformal deformation space $QH(\Gamma_F)$ of a Fuchsian group $\Gamma_F \subset PSL_2(\mathbb{R})$.

In other words, a quasi-Fuchsian manifold is a quotient $\mathbb{H}^3/\Gamma_{qF}$ of $\mathbb{H}^3$ by a discrete finitely generated group $\Gamma_{qF} \subset PSL_2(\mathbb{R})$ of hyperbolic isometries of $\mathbb{H}^3$ such that there is a Fuchsian group Γ_F of isometries of $\mathbb{H}^3$ such that the limit set $\Lambda(\Gamma_{qF}) \subset \partial_\infty \mathbb{H}^3$ of Γ_{qF} is a Jordan curve which can be obtained from the circle $\Lambda(\Gamma_F) \subset \partial_\infty \mathbb{H}^3$ by a quasiconformal deformation of $\partial_\infty \mathbb{H}^3$. The group Γ_{qF} is called *quasi-Fuchsian*.

In geometric terms, a quasi-Fuchsian manifold is a complete hyperbolic manifold homeomorphic to $\mathcal{S} \times \mathbb{R}$, where $\mathcal{S}$ is a closed connected surface of genus at least 2, which contains a convex compact subset.

Let us also recall the A. D. Alexandrov notion of curvature which does not require a metric of a surface to be regular.

Let X be a complete locally compact length space and let $\mathrm{d}_X(\cdot,\cdot)$ stand for the distance between points in X. For a triple of points $p, q, r \in X$ a geodesic triangle $\triangle(pqr)$ is a triple of geodesics joining these three points. For a geodesic triangle $\triangle(pqr) \subset X$ we denote by $\triangle(\tilde{p}\tilde{q}\tilde{r})$ a geodesic triangle sketched in $M^2(K)$ whose corresponding edges have the same lengths as $\triangle(pqr)$.

Definition 1.4 ([22, p. 7]). — X is said to have *curvature bounded below by* K iff every point $x \in X$ has an open neighborhood $U_x \subset X$ such that for every geodesic triangle $\triangle(pqr)$ whose edges are contained entirely in U_x the corresponding geodesic triangle $\triangle(\tilde{p}\tilde{q}\tilde{r})$ sketched in $M^2(K)$ has the following property: for every point $z \in qr$ and for $\tilde{z} \in \tilde{q}\tilde{r}$ with $\mathrm{d}_X(q,z) = \mathrm{d}_{M^2(K)}(\tilde{q},\tilde{z})$ we have

$$\mathrm{d}_X(p,z) \geq \mathrm{d}_{M^2(K)}(\tilde{p},\tilde{z}).$$

In 2016 F. Fillastre, I. Izmestiev, and G. Veronelli [10] proved that for every metric on the torus with curvature bounded from below by -1 in the Alexandrov sense there exists a hyperbolic cusp with convex boundary such that the induced metric on the boundary is the given metric.

Our main goal is to prove the following extension of Theorem 1.5:

Theorem 1.6. — *Let $\mathcal{M}$ be a compact connected 3-manifold with boundary of the type $\mathcal{S} \times [-1,1]$ where $\mathcal{S}$ is a closed connected surface of genus at least 2. Let h be a metric on $\partial\mathcal{M}$ of curvature $K \geq -1$ in the Alexandrov sense. Then there exists a hyperbolic metric g in $\mathcal{M}$ with a convex boundary $\partial\mathcal{M}$ such that the metric induced on $\partial\mathcal{M}$ is h.*

In particular, the following result proved in [23] immediately follows from Theorem 1.6.

Theorem 1.7. — *Let $\mathcal{M}$ be a compact connected 3-manifold with boundary of the type $\mathcal{S} \times [-1,1]$ where $\mathcal{S}$ is a closed connected surface of genus at least 2. Let h be a hyperbolic metric with cone singularities of angle less than 2π on $\partial\mathcal{M}$ such that every singular point of h possesses a neighborhood in $\partial\mathcal{M}$ which does not contain other singular points of h. Then there exists a hyperbolic metric g in $\mathcal{M}$ with a convex boundary $\partial\mathcal{M}$ such that the metric induced on $\partial\mathcal{M}$ is h.*

The idea of the proof of Theorem 1.7 is given in [25].

Theorem 1.7 can also be considered as an analog of Theorem 1.1 for the convex hyperbolic manifolds with polyhedral boundary.

Definition 1.5 ([7]). — A *pleated surface* in a hyperbolic 3-manifold $\mathcal{M}$ is a complete hyperbolic surface $\mathcal{S}$ together with an isometric map $f : \mathcal{S} \to \mathcal{M}$ such that every $s \in \mathcal{S}$ is in the interior of some geodesic arc which is mapped by f to a geodesic arc in $\mathcal{M}$.

A pleated surface resembles a polyhedron in the sense that it has flat faces that meet along edges. Unlike a polyhedron, a pleated surface has no corners, but it may have infinitely many edges that form a lamination.

REMARK 1.8. — The surfaces serving as the connected components of the boundary $\partial\mathcal{M}$ of the manifold $\mathcal{M}$ from the statement of Theorem 1.7, which are equipped by assumption with hyperbolic polyhedral metrics, do not necessarily have to be polyhedra embedded in $\mathcal{M}$: these surfaces can be partially pleated, i.e., the universal covers in $\mathbb{H}^3$ of these surfaces can contain pleated 2-dimensional domains situated between several pairwise nonintersecting geodesics which are also geodesics in $\mathbb{H}^3$.

DEFINITION 1.6 ([16]). — Let $\mathcal{M}$ be the interior of a compact manifold with boundary. A complete hyperbolic metric g on $\mathcal{M}$ is convex co-compact if $\mathcal{M}$ contains a compact subset $\mathcal{K}$ which is convex: any geodesic segment c in $(\mathcal{M}, g)$ with endpoints in $\mathcal{K}$ is contained in $\mathcal{K}$.

In 2002 J.-M. Schlenker [21] proved uniqueness of the metric g in Theorem 1.5. Thus, he obtained

THEOREM 1.9. — *Let M be a compact connected 3-manifold with boundary (different from the solid torus) which admits a complete hyperbolic convex co-compact metric. Let g be a hyperbolic metric on M such that ∂M is C^∞-regular and strictly convex. Then the induced metric I on ∂M has curvature $K > -1$. Each C^∞-regular metric on ∂M with $K > -1$ is induced on ∂M for a unique choice of g.*

It would be natural to conjecture that the metric g in the statements of Theorems 1.6 and 1.7 is unique. The methods used in their demonstration do not presently allow to attack this problem.

At last, recalling that the convex quasi-Fuchsian manifolds are special cases of the convex co-compact manifolds, we can guess that Theorems 1.6 and 1.7 remain valid in the case when $\mathcal{M}$ is a convex co-compact manifold. It would be interesting to verify this hypothesis in the future.

2. Construction of a quasi-Fuchsian manifold containing a compact convex domain with a prescribed Alexandrov metric of curvature $K \geq -1$ on the boundary

A compact connected 3-manifold $\mathcal{M}$ of the type $\mathcal{S} \times [-1, 1]$ from the statement of Theorem 1.6, where $\mathcal{S}$ is a closed connected surface of genus at least 2, can be regarded as a convex compact 3-dimensional domain of an unbounded quasi-Fuchsian manifold $\mathcal{M}^\circ = \mathbb{H}^3/\Gamma_{QF}$ where Γ_{QF} stands for a quasi-Fuchsian group of isometries of hyperbolic space $\mathbb{H}^3$. Note that the boundary $\partial\mathcal{M}$ of such domain $\mathcal{M}$ consists of two distinct locally convex compact 2-surfaces in $\mathcal{M}^\circ$.