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Jonathan Zachhuber

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Bulletin de la Société Mathématique de France
Société Mathématique de France
Institut Henri Poincaré, 11, rue Pierre et Marie Curie
75231 Paris Cedex 05, France
Tél. : (33) 01 44 27 67 99
`bulletin@smf.emath.fr` • `http://smf.emath.fr`

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THE GALOIS ACTION AND A SPIN INVARIANT FOR PRYM-TEICHMÜLLER CURVES IN GENUS 3

by Jonathan Zachhuber

Abstract. — Given a Prym-Teichmüller curve in $\mathcal{M}_3$, this note provides an invariant that sorts the cusp prototypes of Lanneau and Nguyen by component. This can be seen as an analog of McMullen's genus 2 spin invariant, although the source of this invariant is different. Moreover, we describe the Galois action on the cusps of these Teichmüller curves, extending the results of Bouw and Möller in genus 2. We use this to show that the components of the genus 3 Prym-Teichmüller curves are homeomorphic.

Résumé (*L'action galoisienne et un invariant spin pour les courbes de Prym-Teichmüller en genre 3*). — Étant donnée une courbe de Prym–Teichmüller dans $\mathcal{M}_3$, cette note introduit un invariant qui trie par composante les prototypes cusp de Lanneau et Nguyen. Il peut être vu comme l'analogue en genre 3 de l'invariant spin en genre 2 de McMullen, bien que la source de cet invariant soit différente. De plus, nous décrivons l'action de Galois sur les cusps des courbes de Teichmüller, étendant les résultats en genre 2 de Bouw et Möller. Cela nous permet de montrer que les composants des courbes de Prym–Teichmüller en genre 3 sont homéomorphes.

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Jonathan Zachhuber, FB 12 – Institut für Mathematik, Johann Wolfgang Goethe-Universität, Robert-Mayer-Str. 6–8, D-60325 Frankfurt am Main •
E-mail : `zachhuber@math.uni-frankfurt.de`

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1. Introduction

A *Teichmüller curve* is a curve inside the moduli space $\mathcal{M}_g$ of smooth projective genus g curves that is totally geodesic for the Teichmüller metric. Every Teichmüller curve arises as the projection of the $\mathrm{GL}_2^+(\mathbb{R})$ orbit of a flat surface (see section 2 and the references therein for background and definitions). Only a few infinite families of primitive Teichmüller curves are known. McMullen constructed several primitive families in low genera, among them, for every non-square discriminant D, the *Prym-Teichmüller* or *Prym-Weierstraß* curves W_D in genus 3 [4].

This family is fairly well understood. In particular, Möller calculated the Euler characteristic [8], Lanneau and Nguyen enumerated the cusps and connected components [2], and the number and type of orbifold points are determined in [12]. The aim of this note is to complete the classification of the topological components by showing that the connected components of W_D are always homeomorphic.

To be more precise, in [2], Lanneau and Nguyen show that W_D has at most two components for any D and has two components if and only if $D \equiv 1 \bmod 8$.

THEOREM 1.1. — *Let $D \equiv 1 \bmod 8$, which is not a square. Then the two components of W_D are homeomorphic (as orbifolds). In particular, they have the same number of cusps and elliptic points.*

A similar result was obtained by Bouw and Möller [1] for Teichmüller curves in genus 2. Note that a Teichmüller curve is always defined over a number field but is never compact. Both approaches rely on determining the stable curves associated to the cusps of the Teichmüller curve and describing explicitly the Galois action on these cusps. At this point, it is crucial that we are able to determine of a pair of cusps if they lie on the same component or not. In genus 2, Bouw and Möller could use McMullen's spin invariant [3] to achieve this.

However, while Lanneau and Nguyen list prototypes corresponding to the cusps of Prym-Teichmüller curves [2], they do not provide an effective analog of the spin invariant. Here we give such an invariant, which is, moreover, easy to compute.

THEOREM 1.2. — *Let $D \equiv 1 \bmod 8$, which is not a square. Given a cusp prototype $[w, h, t, e, \varepsilon]$ (see section 2), the associated cusp of W_D lies on the component W_D^i if and only if*

$$2i \equiv e + \varepsilon \bmod 4,$$

for $i = 1, 2$.

In section 3, we prove Theorem 1.2 essentially using topological arguments. More precisely, we analyze the intersection pairing on a certain intrinsic subspace of homology with $\mathbb{Z}/2\mathbb{Z}$ coefficients. This is similar to the approach of [3] where the Arf invariant of a quadratic form that was associated to the flat structure was analyzed on such a subspace, but the nature of these subspaces is different (cf. [2, Remark 2.9]). Note also that in genus 3 the two components lie on disjoint Hilbert Modular Surfaces (cf. [8, Proposition 4.6]) and that the $(1, 2)$-polarization of the Prym variety plays a special role in this case, essentially yielding a much more compact formula (cf. [3, Theorem 5.3]).

In section 4, we proceed to give an explicit description of the Galois action on Lanneau and Nguyen's cusp prototypes (Proposition 4.7) and combine this with Theorem 1.2 to show that Galois-conjugate cusps always lie on different components of W_D, thus proving Theorem 1.1.

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2. Cusp Prototypes

A *flat surface* is a pair (X, ω) where X is a compact Riemann surface of genus g and $\omega \in \mathrm{H}^0(X, \omega_X)$ is a holomorphic 1-form on X. Note that X obtains a flat structure away from the zeros of ω via integrating ω and affine shearing of this flat structure gives an action of $\mathrm{GL}_2^+(\mathbb{R})$. A *Teichmüller curve* is a $\mathrm{GL}_2^+(\mathbb{R})$ orbit of a flat surface that projects to an algebraic curve inside the moduli space $\mathcal{M}_g$. See e.g., [7] for background on Teichmüller curves and flat surfaces. Not many families of primitive Teichmüller curves are known; McMullen constructed families in low genera by requiring a factor of the Jacobian of X to admit real multiplication, the (Prym-)Weierstraß curves. We briefly review the construction in genus 3, the case with which we are concerned.

Prym Varieties and Real Multiplication. — Let $D \equiv 0, 1 \bmod 4$ be a (positive) non-square discriminant and denote by $\mathcal{O}_D$ the corresponding order in the real quadratic number field $\mathbb{Q}(\sqrt{D})$. Let X be a genus 3 curve and ρ an involution with X/ρ of genus 1. Then we define the *Prym Variety* $\mathrm{Prym}(X, \rho)$ as the connected component of the identity of $\ker(\mathrm{Jac}(X) \to \mathrm{Jac}(X/\rho))$ and we say that (X, ρ) *admits real multiplication by* $\mathcal{O}_D$ if there exists an injective ring homomorphism $\iota \colon \mathcal{O}_D \to \operatorname{End} \mathrm{Prym}(X, \rho)$, such that

- every endomorphism $\iota(s)$ is self-adjoint with respect to the intersection pairing on H_1, and
- ι cannot be extended to any $\mathcal{O}_{D'} \supset \mathcal{O}_D$.

In other words, the ρ-anti-invariant part $\mathrm{H}_1(X, \mathbb{Z})^-$ of the homology admits a symplectic $\mathcal{O}_D$-module structure and $\mathcal{O}_D$ is maximal in this respect.