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POINCARÉ DUALITY AND COMMUTATIVE DIFFERENTIAL GRADED ALGEBRAS

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Abstract. – We prove that every commutative differential graded algebra whose cohomology is a simply-connected Poincaré duality algebra is quasi-isomorphic to one whose underlying algebra is simply-connected and satisfies Poincaré duality in the same dimension. This has applications in rational homotopy, giving Poincaré duality at the cochain level, which is of interest in particular in the study of configuration spaces and in string topology.

Résumé. – Nous démontrons que toute algèbre différentielle graduée commutative (ADGC) dont la cohomologie est une algèbre simplement connexe à dualité de Poincaré est quasi-isomorphe à une ADGC dont l'algèbre sous-jacente est à dualité de Poincaré dans la même dimension. Ce résultat a des applications en théorie de l'homotopie rationnelle, permettant d'obtenir la dualité de Poincaré au niveau des cochaînes, entre autres dans l'étude des espaces de configurations et en topologie des cordes.

1. Introduction

The first motivation for the main result of this paper comes from rational homotopy theory. Recall that Sullivan [16] has constructed a contravariant functor

$$\mathrm{A_{PL}}\colon\ \mathrm{Top}\to\mathrm{CDGA}_{\mathbb{Q}}$$

from the category of topological space to the category of commutative differential graded algebras over the field $\mathbb{Q}$ (see Section 2 for the definition). The main feature of $\mathrm{A_{PL}}$ is that when X is a simply-connected space with rational homology of finite type, then the rational homotopy type of X is completely encoded in any CDGA (A,d) weakly equivalent to $\mathrm{A_{PL}}(X)$. By *weakly equivalent* we mean that (A,d) and $\mathrm{A_{PL}}(X)$ are connected by a zig-zag of CDGA morphisms inducing isomorphisms in homology, or *quasi-isomorphisms* for short,

$$(A,d)\xleftarrow{\simeq}\dots\xrightarrow{\simeq}\mathrm{A_{PL}}(X).$$

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We then say that (A, d) is a *CDGA-model* of X (see [4] for a complete exposition of this theory). We are particularly interested in the case when X is a simply-connected closed manifold of dimension n, since then $\mathrm{H}^*(A, d)$ is a simply-connected Poincaré duality algebra of dimension n (a graded algebra A is said to be *simply-connected* if A^0 is isomorphic to the ground field and $A^1 = 0$; see also Definition 2.1).

Our main result is the following:

THEOREM 1.1. – *Let* $\mathbf{k}$ *be a field of any characteristic and let* (A, d) *be a CDGA over* $\mathbf{k}$ *such that* $\mathrm{H}^*(A, d)$ *is a simply-connected Poincaré duality algebra in dimension* n*. Then there exists a CDGA* (A', d') *weakly equivalent to* (A, d) *and such that* A' *is a simply-connected algebra satisfying Poincaré duality in dimension* n*.*

Our theorem was conjectured by Steve Halperin over 20 years ago. The A' of the theorem is called a *differential Poincaré duality algebra* or *Poincaré duality CDGA* (Definition 2.2). In particular any simply-connected closed manifold admits a Poincaré duality CDGA-model.

Notice that the theorem is even valid for a field of non-zero characteristic. Also our proof is very constructive: Starting from a finite-dimensional CDGA (A, d), it shows how to compute explicitly a weakly equivalent differential Poincaré duality algebra (A', d'). We will also prove in the last section that under some extra connectivity hypotheses, any two such weakly equivalent differential Poincaré duality algebra can be connected by a zig-zag of quasi-isomorphisms between differential Poincaré duality algebras.

Aubry, Lemaire, and Halperin [1] and Lambrechts [9, p.158] prove the main result of this paper in some special cases. Also in [15] Stasheff proves some chain level results about Poincaré duality using Quillen models. An error in Stasheff's paper was corrected in [1].

Before giving the idea of the proof of Theorem 1.1, we describe a few applications of this result.

1.1. Applications

There should be many applications of this result to constructions in rational homotopy theory involving Poincaré duality spaces. We consider here two: The first is to the study of configuration spaces over a closed manifold, and the second to string topology.

Our first application is to the determination of the rational homotopy type of the configuration space

$$F(M, k) := \{(x_1, \ldots, x_k) \in M^k : x_i \neq x_j \text{ for } i \neq j\}$$

of k points in a closed manifold M of dimension n. When $k = 2$ and M is 2-connected, we showed in [10, Theorem 1.2] that if A is a Poincaré duality CDGA-model of M then a CDGA-model of $F(M, 2)$ is given by

$$A \otimes A/(\Delta) \tag{1.1}$$

where (Δ) is the differential ideal in $A \otimes A$ generated by the so-called *diagonal class* $\Delta \in (A \otimes A)^n$.

For $k \geq 2$ we have constructed in [11] an explicit CDGA

$$F(A, k)$$

generalizing (1.1) and which is an $A^{\otimes k}$-*DG-module model* of $F(M,k)$. Poincaré duality of the CDGA A is an essential ingredient in the construction of $F(A,k)$. If M is a smooth complex projective variety then we can use $H^*(M)$ as a model for M, and in this case $F(H^*(M),k)$ is exactly the model of Kriz and Fulton-Mac Pherson [8][7] for $F(M,k)$. However we do not know in general if $F(A,k)$ is also a CDGA-model although it seems to be the natural candidate.

A second application is to string topology, a new field created by Chas and Sullivan [3]. They constructed a product, a bracket and a Δ operator on the homology of the free loop space $LM = M^{S^1}$ of a closed simply-connected manifold M, that turned it into Gerstenhaber algebra and even a BV algebra. On the Hochschild cohomology $\mathrm{HH}^*(A,A)$ of a (differential graded) algebra A, there are the classical cup product and Gerstenhaber bracket, and Tradler [17] showed that for $A = C^*(M)$ there is also a Δ operator on Hochschild homology making it into a BV algebra. Menichi [12] later reproved this result and showed that the Δ can be taken to be the dual of the Connes boundary operator. Recently Felix and Thomas [5] have shown that over the rationals the Chas-Sullivan BV structure on the homology of LM is isomorphic to the BV structure on $\mathrm{HH}^*(C^*(M), C^*(M))$. Their proof uses the main result of this paper. Yang [18] also uses our results to give explicit formulas for the BV-algebra structure on Hochschild cohomology.

1.2. Idea of proof

The proof is completely constructive. We start with a CDGA (A,d) and an orientation $\epsilon\colon A^n \to \mathbf{k}$ (Definition 2.3). We consider the pairing at the chain level

$$\phi\colon A^k \otimes A^{n-k} \to \mathbf{k},\ a \otimes b \mapsto \epsilon(ab).$$

We may assume that ϕ induces a non degenerate bilinear form on cohomology making $\mathrm{H}^*(A)$ into a Poincaré duality algebra. The problem is that ϕ itself may be degenerate; there may be some orphan elements (see Definition 3.1) a with $\epsilon(ab) = 0$ for all b. Quotienting out by the orphans $\mathcal{O}$ we get a differential Poincaré duality algebra $A/\mathcal{O}$, and a map $f\colon A \to A/\mathcal{O}$ (Proposition 3.3). With this observation the heart of the proof begins.

Now the problem is that f might not be a quasi-isomorphism - this happens whenever $\mathrm{H}^*(\mathcal{O}) \neq 0$. The solution is to add generators to A to get a quasi-isomorphic algebra $\hat{A}$ with better properties. An important observation is that $\mathrm{H}^*(\mathcal{O})$ satisfies a kind of Poincaré duality so it is enough to eliminate $\mathrm{H}^*(\mathcal{O})$ starting from about half of the dimension and working up from there. In some sense we perform something akin to surgery by eliminating the cohomology of the orphans in high dimensions and having the lower dimensional cohomology naturally disappear at the same time. In the middle dimension, the extra generators have the effect of turning orphans which represent homology classes into non orphans. In higher dimensions some of the new generators become orphans whose boundaries kill elements of $\mathrm{H}^*(\mathcal{O})$. In both cases the construction introduces no new orphan homology between the middle dimension and the dimension where the elements of $\mathrm{H}^*(\mathcal{O})$ are killed. This together with the duality in $\mathrm{H}^*(\mathcal{O})$ is enough to get an inductive proof of Theorem 1.1.

2. Some terminology

Just for the record we introduce the terms CDGA, Poincaré duality algebra and differential Poincaré duality algebra.

We fix once and for all a ground field $\mathbf{k}$ of any characteristic. So tensor product, algebras, etc., will always be over that field. A commutative differential graded algebra, or CDGA, (A, d) is a non-negatively graded commutative algebra, together with a differential d of degree $+1$. If an element $a \in A$ is in degree n, we write $|a| = n$. The set of elements of degree n in A is denoted by A^n. Since A is graded commutative we have the formula $ab = (-1)^{|a||b|}ba$ and $a^2 = 0$ when $|a|$ is odd, including when $\mathbf{k}$ is of characteristic 2. Also d satisfies the graded Leibnitz formula $d(ab) = (da)b + (-1)^{|a|}adb$. CDGA over the rationals are of particular interest since they are models of rational homotopy theory. For more details see [4].

Convention. – *All of the CDGA we consider in this paper will be connected, in other words $A^0 = \mathbf{k}$, and of finite type.*

Note that every simply connected CW-complex of finite type admits such a CDGA model of its rational homotopy type.

Poincaré duality is defined as follows:

Definition 2.1. – An *oriented Poincaré duality algebra* of dimension n is a pair (A, ϵ) such that A is a connected graded commutative algebra and $\epsilon\colon A^n \to \mathbf{k}$ is a linear map such that the induced bilinear forms

$$A^k \otimes A^{n-k} \to \mathbf{k},\ a \otimes b \mapsto \epsilon(ab)$$

are non-degenerate.

The following definition comes from [10]:

Definition 2.2. – An *oriented differential Poincaré duality algebra* or *oriented Poincaré CDGA* is a triple (A, d, ϵ) such that

(i) (A, d) is a CDGA,
(ii) (A, ϵ) is an oriented Poincaré duality algebra,
(iii) $\epsilon(dA) = 0$.

An oriented differential Poincaré duality algebra is essentially a CDGA whose underlying algebra satisfies Poincaré duality. The condition $\epsilon(dA) = 0$ is equivalent to $\mathrm{H}^*(A, d)$ being a Poincaré duality algebra in the same dimension [10, Proposition 4.8].

For convenience we make the following:

Definition 2.3. – An *orientation* of a CGDA (A, d) is a linear map

$$\epsilon\colon A^n \to \mathbf{k}$$

such that $\epsilon(dA^{n-1}) = 0$ and there exists a cocycle $\mu \in A^n \cap \ker d$ with $\epsilon(\mu) = 1$.