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PSEUDO-ABELIAN VARIETIES

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ABSTRACT. – Chevalley's theorem states that every smooth connected algebraic group over a perfect field is an extension of an abelian variety by a smooth connected affine group. That fails when the base field is not perfect. We define a pseudo-abelian variety over an arbitrary field k to be a smooth connected k-group in which every smooth connected affine normal k-subgroup is trivial. This gives a new point of view on the classification of algebraic groups: every smooth connected group over a field is an extension of a pseudo-abelian variety by a smooth connected affine group, in a unique way.

We work out much of the structure of pseudo-abelian varieties. These groups are closely related to unipotent groups in characteristic p and to pseudo-reductive groups as studied by Tits and Conrad-Gabber-Prasad. Many properties of abelian varieties such as the Mordell-Weil theorem extend to pseudo-abelian varieties. Finally, we conjecture a description of $\mathrm{Ext}^2(\mathbf{G}_a, \mathbf{G}_m)$ over any field by generators and relations, in the spirit of the Milnor conjecture.

RÉSUMÉ. – Le théorème de Chevalley affirme que tout groupe algébrique lisse connexe sur un corps parfait est une extension d'une variété abélienne par un groupe affine lisse connexe. Cela n'est plus vrai lorsque le corps de base n'est pas parfait. Nous définissons une variété pseudo-abélienne sur un corps arbitraire k en tant que k-groupe lisse connexe dans lequel tous les k-sous-groupes lisses connexes affines distingués sont triviaux. Cela donne un nouveau point de vue sur la classification des groupes algébriques: tout groupe lisse connexe sur un corps est une extension, faite de manière unique, d'une variété pseudo-abélienne par un groupe lisse connexe affine.

Nous déterminons une grande partie de la structure des variétés pseudo-abéliennes. Ces groupes sont étroitement liés aux groupes unipotents en caractéristique p et aux groupes pseudo-réductifs étudiés par Tits et Conrad-Gabber-Prasad. Plusieurs propriétés des variétés abéliennes (comme le théorème de Mordell-Weil) s'étendent aux variétés pseudo-abéliennes. Enfin, nous conjecturons une description de $\mathrm{Ext}^2(\mathbf{G}_a, \mathbf{G}_m)$ sur n'importe quel corps par générateurs et relations, dans l'esprit de la conjecture de Milnor.

The theory of algebraic groups is divided into two parts with very different flavors: affine algebraic groups (which can be viewed as matrix groups) and abelian varieties. Concentrating on these two types of groups makes sense in view of Chevalley's theorem: for a perfect field k,

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every smooth connected k-group G is an extension of an abelian variety A by a smooth connected affine k-group N [9, 10]:

$$1 \to N \to G \to A \to 1.$$

But Chevalley's theorem fails over every imperfect field. What can be said about the structure of a smooth connected algebraic group over an arbitrary field k? (Group schemes which are neither affine nor proper come up naturally, for example as the automorphism group scheme or the Picard scheme of a projective variety over k. Groups over an imperfect field such as the rational function field $\mathbf{F}_p(t)$ arise geometrically as the generic fiber of a family of groups in characteristic p.)

One substitute for Chevalley's theorem that works over an arbitrary field is that every connected group scheme (always assumed to be of finite type) over a field k is an extension of an abelian variety by a connected affine group scheme, not uniquely [26, Lemme IX.2.7]. But when this result is applied to a smooth k-group, the affine subgroup scheme may have to be non-smooth. And it is desirable to understand the structure of smooth k-groups as far as possible without bringing in the complexities of arbitrary k-group schemes. To see how far group schemes can be from being smooth, note that every group scheme G of finite type over a field k has a unique maximal smooth closed k-subgroup [12, Lemma C.4.1], but (for k imperfect) that subgroup can be trivial even when G has positive dimension. (A simple example is the group scheme $G = \{(x,y) \in (\mathbf{G}_a)^2 : x^p = ty^p\}$ for $t \in k$ not a pth power, where p is the characteristic of k. The dimension of G is 1, but the maximal smooth k-subgroup of G is the trivial group.)

Brion gave a useful structure theorem for smooth k-groups by putting the smooth affine group "on top". Namely, for any field k of positive characteristic, every smooth connected k-group is a central extension of a smooth connected affine k-group by a semi-abelian variety (an extension of an abelian variety by a torus) [8, Proposition 2.2]. (Another proof was given by C. Sancho de Salas and F. Sancho de Salas [29].) One can still ask what substitute for Chevalley's theorem works over arbitrary fields, with the smooth affine group "on the bottom". We can gain inspiration from Tits's theory of pseudo-reductive groups [35, 36], developed by Conrad-Gabber-Prasad [12]. By definition, a pseudo-reductive group over a field k is a smooth connected affine k-group G such that every smooth connected unipotent normal k-subgroup of G is trivial. That suggests the definition:

Definition 0.1. – A *pseudo-abelian variety* over a field k is a smooth connected k-group G such that every smooth connected affine normal k-subgroup of G is trivial.

It is immediate that every smooth connected group over a field k is an extension of a pseudo-abelian variety by a smooth connected affine group over k, in a unique way. Whether this is useful depends on what can be said about the structure of pseudo-abelian varieties. Chevalley's theorem implies that a pseudo-abelian variety over a perfect field is simply an abelian variety.

Over any imperfect field, Raynaud constructed pseudo-abelian varieties which are not abelian varieties [14, Exp. XVII, App. III, Prop. 5.1]. Namely, for any finite purely inseparable extension l/k and any abelian variety B over l, the Weil restriction $R_{l/k}B$ is a pseudo-abelian variety, and it is not an abelian variety if $l \neq k$ and $B \neq 0$. (Weil restriction produces a

k-scheme $R_{l/k}B$ whose set of k-rational points is equal to the set of l-rational points of B.) Indeed, over an algebraic closure $\overline{k}$ of l, $R_{l/k}B$ becomes an extension of $B_{\overline{k}}$ by a smooth unipotent group of dimension $([l\colon k]-1)\dim(B)$, and so $R_{l/k}B$ is not an abelian variety. (This example shows that the notion of a pseudo-abelian variety is not geometric, in the sense that it is not preserved by arbitrary field extensions. It is preserved by separable field extensions, however.)

One main result of this paper is that every pseudo-abelian variety over a field k is *commutative*, and every pseudo-abelian variety is an extension of a smooth connected commutative unipotent k-group by an abelian variety (Theorem 2.1). In this sense, pseudo-abelian varieties are reasonably close to abelian varieties. So it is a meaningful generalization of Chevalley's theorem to say that every smooth connected group over a field k is an extension of a pseudo-abelian variety by a smooth connected affine group over k.

One can expect many properties of abelian varieties to extend to pseudo-abelian varieties. For example, the Mordell-Weil theorem holds for pseudo-abelian varieties (Proposition 4.1). Like abelian varieties, pseudo-abelian varieties can be characterized among all smooth connected groups G over a field k without using the group structure, in fact using only the birational equivalence class of G over k: G is a pseudo-abelian variety if and only if G is not "smoothly uniruled" (Theorem 5.1).

The other main result is that, over an imperfect field of characteristic p, *every* smooth connected commutative group of exponent p occurs as the unipotent quotient of some pseudo-abelian variety (Corollaries 6.5 and 7.3). Over an imperfect field, smooth commutative unipotent groups form a rich family, studied by Serre, Tits, Oesterlé, and others over the past 50 years [17], [25], [12, Appendix B]. So there are far more pseudo-abelian varieties (over any imperfect field) than the initial examples, Weil restrictions of abelian varieties.

Lemma 8.1 gives a precise relation between the structure of certain pseudo-abelian varieties and the (largely unknown) structure of commutative pseudo-reductive groups. We prove some new results about commutative pseudo-reductive groups. First, a smooth connected unipotent group of dimension 1 over a field k occurs as the unipotent quotient of some commutative pseudo-reductive group if and only if it is not isomorphic to the additive group $\mathbf{G}_a$ over k (Corollary 9.5). But an analogous statement fails in dimension 2 (Example 9.7). The proofs include some tools for computing the invariants $\mathrm{Ext}^1(U,\mathbf{G}_m)$ and $\mathrm{Pic}(U)$ of a unipotent group U. Finally, Question 7.4 conjectures a calculation of $\mathrm{Ext}^2(\mathbf{G}_a,\mathbf{G}_m)$ over any field by generators and relations, in the spirit of the Milnor conjecture. Question 9.11 attempts to describe the commutative pseudo-reductive groups over 1-dimensional fields.

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1. Notation

A *variety* over a field k means an integral separated scheme of finite type over k. Let k be a field with algebraic closure $\overline{k}$ and separable closure k_s. A field extension F of k (not necessarily algebraic) is *separable* if the ring $F\otimes_k\overline{k}$ contains no nilpotent elements other than zero. For example, the function field of a variety X over k is separable over k if and only if the smooth locus of X over k is nonempty [5, Section X.7, Theorem 1, Remark 2, Corollary 2].

We use the convention that a connected topological space is nonempty.

A group scheme over a field k is *unipotent* if it is isomorphic to a k-subgroup scheme of the group of strictly upper triangular matrices in $GL(n)$ for some n (see [14, Théorème XVII.3.5] for several equivalent conditions). Being unipotent is a geometric property, meaning that it does not change under field extensions of k. Unipotence passes to subgroup schemes, quotient groups, and group extensions.

We write $\mathbf{G}_a$ for the additive group. Over a field k of characteristic $p > 0$, we write α_p for the k-group scheme $\{x \in \mathbf{G}_a : x^p = 0\}$. A group scheme over k is unipotent if and only if it has a composition series with successive quotients isomorphic to α_p, $\mathbf{G}_a$, or k-forms of $(\mathbf{Z}/p)^r$ [14, Théorème XVII.3.5].

Tits defined a smooth connected unipotent group over a field k to be k-*wound* if it does not contain $\mathbf{G}_a$ as a subgroup over k. When k has characteristic $p > 0$, a smooth connected commutative k-group of exponent p can be described in a unique way as an extension of a k-wound group by a subgroup isomorphic to $(\mathbf{G}_a)^n$ for some $n \geq 0$ [12, Theorem B.3.4]. Over a perfect field, a k-wound group is trivial. An example of a nontrivial k-wound group is the smooth connected subgroup $\{(x, y) : y^p = x - tx^p\}$ of $(\mathbf{G}_a)^2$ for any $t \in k - k^p$, discussed in Example 9.6.

Over an imperfect field k of characteristic p, there are many smooth connected commutative groups of exponent p (although they all become isomorphic to $(\mathbf{G}_a)^n$ over the algebraic closure of k). One striking phenomenon is that some of these groups are k-rational varieties, while others contain no k-rational curves [17, Theorem 6.9.2], [25, Theorem VI.3.1]. Explicitly, define a *p-polynomial* to be a polynomial with coefficients in k such that every monomial in f is a single variable raised to some power of p. Then every smooth connected commutative k-group of exponent p and dimension n is isomorphic to the subgroup of $(\mathbf{G}_a)^{n+1}$ defined by some p-polynomial f with nonzero degree-1 part [25, Proposition V.4.1], [12, Proposition B.1.13].

A smooth connected affine group G over a field k is *pseudo-reductive* if every smooth connected unipotent normal k-subgroup of G is trivial. The stronger property that G is *reductive* means that every smooth connected unipotent normal subgroup of $G_{\overline{k}}$ is trivial.

We write $\mathbf{G}_m$ for the multiplicative group over k. For each positive integer n, the k-group scheme $\{x \in \mathbf{G}_m : x^n = 1\}$ of nth roots of unity is called μ_n. A k-group scheme M is of *multiplicative type* if it is the dual of some $\mathrm{Gal}(k_s/k)$-module L which is finitely generated as an abelian group, meaning that $M = \mathrm{Spec}(k_s[L])^{\mathrm{Gal}(k_s/k)}$ [14, Proposition X.1.4]. Dualizing the surjection $L \to L/L_{\mathrm{tors}}$ shows that every k-group scheme M of multiplicative type contains a k-torus T with M/T finite. (Explicitly, T is the identity component of M with reduced scheme structure.)

2. Structure of pseudo-abelian varieties

THEOREM 2.1. – *Every pseudo-abelian variety E over a field k is commutative. Moreover, E is in a unique way an extension of a smooth connected commutative unipotent k-group U by*