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POISSON SUSPENSIONS AND SUSHIS

BY ÉLISE JANVRESSE, EMMANUEL ROY AND THIERRY DE LA RUE

ABSTRACT. – In this paper, we prove that ergodic point processes with moments of all orders, driven by particular infinite measure preserving transformations, have to be a superposition of shifted Poisson processes. This rigidity result has a lot of implications in terms of joining and disjointness for the corresponding Poisson suspension. In particular, we prove that its ergodic self-joinings are Poisson joinings, which provides an analog, in the Poissonian context, of the GAG property for Gaussian dynamical systems.

RÉSUMÉ. – Dans cet article, nous démontrons qu'un processus ponctuel ergodique avec des moments de tous ordres, dirigé par une transformation préservant une mesure infinie qui vérifie certaines propriétés, est nécessairement une superposition de processus de Poisson décalés. Ce résultat de rigidité a de nombreuses implications en termes de couplages et de disjonction pour la suspension de Poisson associée. En particulier, nous démontrons que ses autocouplages ergodiques sont des couplages poissoniens, obtenant ainsi un analogue, dans le contexte poissonien, de la propriété GAG des systèmes dynamiques gaussiens.

1. Introduction

Central to probability theory are Gaussian and Poisson distributions. In ergodic theory, they both play a particular role through canonical constructions we briefly recall:

- Starting from a positive and finite symmetric Borel measure σ on $\mathbb{T}$, there exists a unique centered stationary real-valued Gaussian process $\{X_n\}_{n\in\mathbb{Z}}$ whose coordinates admit σ as spectral measure, that is

$$\mathbb{E}\left[X_0 X_n\right] = \widehat{\sigma}\left(n\right).$$

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- Starting from a σ-finite dynamical system $(X, \mathscr{A}, \mu, T)$, we can build the Poisson suspension $\left(X^*, \mathscr{A}^*, \mu^*, T_*\right)$, which is the canonical space $\left(X^*, \mathscr{A}^*, \mu^*\right)$ of the Poisson point process of intensity μ, enriched by the transformation

$$T_* (\xi) := \xi \circ T^{-1}$$

(see below for a precise definition).

A striking theorem due to Foiaş and Strătilă (see [9]) states that some measures σ on $\mathbb{T}$, if appearing as spectral measure of some ergodic stationary process, force the process to be Gaussian. This was considerably developed later (see [18] in particular) and lead to some remarkable results.

In this paper, we obtain a Poisson counterpart of Foiaş-Strătilă theorem. We prove that some ergodic infinite measure preserving transformation, taken as base system of an ergodic invariant point process with moments of all orders, force the latter to be a superposition of shifted Poisson point processes.

Notations. – For any set J, we denote by $\#J$ the cardinality of J. If φ is any measurable map from $(X, \mathscr{A})$ to $(Y, \mathscr{B})$, and if m is a measure on $(X, \mathscr{A})$, we denote by $\varphi_*(m)$ the pushforward measure of m by φ.

1.1. Random measures and point processes

Let X be a complete separable metric space and $\mathscr{A}$ be its Borel σ-algebra. Define $\widetilde{X}$ to be the space of *boundedly finite measures* on $(X, \mathscr{A})$, that is to say measures giving finite mass to any bounded Borel subset of X. We refer to [4] for the topological properties of $\widetilde{X}$. In particular, $\widetilde{X}$ can be turned into a complete separable metric space, and its Borel σ-algebra $\widetilde{\mathscr{A}}$ is generated by the maps $\widetilde{X} \ni \xi \mapsto \xi(A) \in \mathbb{R}_+ \cup \{+\infty\}$ for bounded $A \in \mathscr{A}$.

Let $X^* \subset \widetilde{X}$ be the subspace of simple counting measures, i.e., whose elements are of the form

$$\xi = \sum_{i \in I} \delta_{x_i},$$

where I is at most countable, and $x_i \neq x_j$ whenever $i \neq j$. Because we restrict ourselves to boundedly finite measures, any bounded subset $A \subset X$ contains finitely many points of the family $\{x_i\}_i$. Conversely, any countable family of points satisfying this property defines a measure $\xi \in X^*$ by the above formula. We define $\mathscr{A}^*$ as the restriction to X^* of $\widetilde{\mathscr{A}}$.

In the paper, we consider a boundedly finite measure μ on X and an invertible transformation T on X preserving μ. We assume that $\mu(X) = \infty$ and that $(X, \mathscr{A}, \mu, T)$ is conservative and ergodic. This implies in particular that μ is continuous.

Given the map T, for any σ-finite measure ξ, we define $T_*(\xi)$ as the pushforward measure of ξ by T. In particular, if $\xi = \sum_{i \in I} \delta_{x_i}$,

$$T_* (\xi) = \sum_{i \in I} \delta_{T(x_i)}.$$

Observe that, even if $\xi \in X^*$, $T_*(\xi)$ is not necessarily in X^* (the property of bounded finiteness may be lost by the action of T). Nevertheless, one can consider the smaller space $\bigcap_{n \in \mathbb{Z}} T_*^{-n} X^*$, on which T_* is a bijective transformation. If m is a probability measure on X^* which is concentrated on this smaller space, then it makes sense to say that m is invariant by T_*. If m is such a T_*-invariant probability measure, then $T_*(\xi) \in X^*$ for m-almost all $\xi \in X^*$, and $(X^*, \mathscr{A}^*, m, T_*)$ is an invertible, probability preserving dynamical system. (The same remark holds if we replace X^* by $\widetilde{X}$.)

We call *point process on* X any random variable N defined on some probability space $(\Omega, \mathscr{F}, \mathbb{P})$ taking values in $\left(X^*, \mathscr{A}^*\right)$. In this case, for $\omega \in \Omega$, $N(\omega)$ can be viewed as a (random) set of points in X, and "$x \in N(\omega)$" means $N(\omega)(\{x\}) = 1$. As usual in probability theory, we will often omit ω in the formulas.

Definition 1.1. – *Let* $(\Omega, \mathscr{F}, \mathbb{P})$ *be a probability space, endowed with a measure preserving invertible transformation* S. *A* T-point process *defined on* $(\Omega, \mathscr{F}, \mathbb{P}, S)$ *is a point process* $N : \Omega \to X^*$, *such that*

– *for any set* $A \in \mathscr{A}$, $N(\omega)(A) = 0$ *for* $\mathbb{P}$*-almost all* ω *whenever* $\mu(A) = 0$;
– *for* $\mathbb{P}$*-almost all* ω, *for any set* $A \in \mathscr{A}$, $N(S\omega)(A) = N(\omega)\left(T^{-1}A\right)$.

Thus, a T-point process N implements a factor relationship between the dynamical systems $(\Omega, \mathscr{F}, \mathbb{P}, S)$ and $\left(X^*, \mathscr{A}^*, m, T_*\right)$ where m is the pushforward measure of $\mathbb{P}$ by N.

Observe that the formula $A \in \mathscr{A} \mapsto \mathbb{E}[N(A)]$ defines a T-invariant measure which is absolutely continuous with respect to μ. It is called the *intensity* of N and as soon as it is σ-finite, by ergodicity of $(X, \mathscr{A}, \mu, T)$, it is a multiple of μ:

$$\mathbb{E}[N(\cdot)] = \alpha\mu(\cdot)$$

for some $\alpha > 0$. In this case, we will say that N is *integrable*. More generally, setting

$$\mathscr{A}_f := \{A \in \mathscr{A},\ 0 < \mu(A) < +\infty\},$$

we have:

Definition 1.2. – *A* T*-point process* N *on* $(\Omega, \mathscr{F}, \mathbb{P}, S)$ *is said to have* moments of order $n \geq 1$ *if, for all* $A \in \mathscr{A}_f$, $\mathbb{E}\left[(N(A))^n\right] < +\infty$. *In this case, for* $k \leq n$, *the formula*

$$M_k^N(A_1 \times \cdots \times A_k) := \mathbb{E}[N(A_1) \times \cdots \times N(A_k)]$$

defines a boundedly finite $T \times \cdots \times T$*-invariant measure* M_k^N *on* $\left(X^k, \mathscr{A}^{\otimes k}\right)$ *called the* k-order moment measure.

A T*-point process with moments of order* 2 *is said to be* square integrable.

1.2. Poisson point process and SuShis

The most important T-point processes are Poisson point processes, let us recall their definition.

Definition 1.3. – *A random variable* N *with values in* $(X^*, \mathscr{A}^*)$ *is a* Poisson point process of intensity μ *if for any* $k \geq 1$, *for any mutually disjoint sets* $A_1, \ldots, A_k \in \mathscr{A}_f$, *the random variables* $N(A_1), \ldots, N(A_k)$ *are independent and Poisson distributed with respective parameters* $\mu(A_1), \ldots, \mu(A_k)$.

Such a process always exists, and its distribution μ^* on X^* is uniquely determined by the preceding conditions. Since T preserves μ, one easily checks that T_* preserves μ^*.

Definition 1.4. – *The probability-preserving dynamical system* $\left(X^*, \mathscr{A}^*, \mu^*, T_*\right)$ *is called the* Poisson suspension *over the base* $(X, \mathscr{A}, \mu, T)$.