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NON-AFFINE LANDAU-GINZBURG MODELS AND INTERSECTION COHOMOLOGY

BY Thomas REICHELT AND Christian SEVENHECK

Abstract. – We construct Landau-Ginzburg models for numerically effective complete intersections in toric manifolds as partial compactifications of families of Laurent polynomials. We show a mirror statement saying that the quantum $\mathscr{D}$-module of the ambient part of the cohomology of the submanifold is isomorphic to an intersection cohomology $\mathscr{D}$-module defined from this partial compactification and we deduce Hodge properties of these differential systems.

Résumé. – Nous construisons un modèle de Landau-Ginzburg pour les intersections complètes numériquement effectives dans les variétés toriques lisses. Il s'agit de compactifications partielles de familles de polynômes de Laurent. Nous démontrons un théorème de symétrie miroir qui exprime le $\mathscr{D}$-module quantique de la partie ambiante de la cohomologie de la sous-variété comme un $\mathscr{D}$-module de cohomologie d'intersection défini par cette compactification partielle. Nous en déduisons des propriétés de Hodge de ces systèmes différentiels.

1. Introduction

The aim of this paper is the construction of a mirror model for complete intersections in smooth toric varieties. We consider the case where these subvarieties have a numerically effective anticanonical bundle. This includes in particular toric Fano manifolds, whose mirror is usually described by oscillating integrals defined by a family of Laurent polynomials and also the most prominent and classical example of mirror symmetry, namely, that of Calabi-Yau hypersurfaces in toric Fano manifolds. Here the mirror is a family of Calabi-Yau manifolds and the mirror correspondence involves the variation of Hodge structures defined by this family. One interesting feature of our results is that these apparently rather different situations occur as special cases of a general mirror construction, called *non-affine Landau-Ginzburg model*.

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It is well-known that quantum cohomology theories admit expressions in terms of certain differential systems, called quantum $\mathscr{D}$-modules. This yields a convenient framework in which mirror symmetry is stated as an equivalence of such systems. Moreover, Hodge theoretic aspects of mirror correspondences can be incorporated using the machinery of (mixed) Hodge modules. However, quantum $\mathscr{D}$-modules have usually irregular singularities, except in the Calabi-Yau case. In our mirror construction, this corresponds to the fact that we let the Fourier-Laplace functor act on various regular $\mathscr{D}$-modules obtained from the Landau-Ginzburg model.

The quantum cohomology of a smooth complete intersection (which in our case is given as the zero locus of a generic section of a vector bundle) can be computed using the so-called *Euler-twisted Gromov-Witten invariants*. Basically, these are integrals over moduli spaces of stable maps of pull-backs of cohomology classes on the variety **and** of the Euler class of the vector bundle. It is well known (see [38, 42, 21] and also [31] as well as [39] for more recent accounts) that the *ambient part* of the quantum cohomology of the subvariety (consisting of those classes which are induced from cohomology classes of the ambient variety), is given as a quotient of the Euler-twisted quantum cohomology.

From the combinatorial toric data of this vector bundle, we construct in a rather straightforward manner an *affine Landau-Ginzburg model*, which is a family of Laurent polynomials. The Euler-twisted quantum $\mathscr{D}$-module (which encodes the above mentioned Euler-twisted Gromov-Witten invariants) can then be shown to be isomorphic a certain proper FL-transformed *Gauß-Manin system*, namely, the Fourier-Laplace transformation of the top cohomology group of the compactly supported direct image complex (in the sense of $\mathscr{D}$-modules) of this affine Landau-Ginzburg model. On the other hand we show that the $Euler^{-1}$-twisted quantum $\mathscr{D}$-module which encodes the so-called local Gromov-Witten invariants is isomorphic to the usual FL-transformed Gauß-Manin system.

The actual non-affine Landau Ginzburg model is constructed by a certain partial compactification of the affine one, which yields a family of projective varieties. Our main result is Theorem 6.13 (which also contains the above mirror statements on twisted resp. local quantum $\mathscr{D}$-modules), it states that the *ambient quantum $\mathscr{D}$-module* is isomorphic to a Fourier-Laplace transform of the direct image of the intersection cohomology $\mathscr{D}$-module of the total space of this family, notice that this total space is usually not smooth.

One of the big advantages of using this singular variety together with the intersection cohomology $\mathscr{D}$-module is the fact that we do not need any kind of resolutions. In particular, we do not need to construct (or suppose the existence of) crepant resolutions like in [2]. Notice also that [31] discusses Landau-Ginzburg models of a more special class of subvarieties in toric orbifolds (the so-called nef partitions). In that paper, a mirror statement is shown in terms of A- resp. B-periods, but this construction needs a hypothesis on the smoothness of a certain complete intersection (given as the intersection of fibres of several Laurent polynomials, see Section 5.2 of loc.cit.). Some more remarks on the nef-partition model and how it relates to our construction can be found in Section 1.5 below.

We will show that the direct image of the intersection cohomology $\mathscr{D}$-module of the total space is itself (modulo some irrelevant free $\mathscr{O}$-modules) an intersection cohomology $\mathscr{D}$-module with respect to a local system measuring the intersection cohomology of the fibers of the projective family. An important point in our paper is that this intersection cohomology $\mathscr{D}$-module admits a hypergeometric description, that is, it can be derived from so-called GKZ-systems (as defined and studied by Gelfand', Kapranov and Zelevinsky). More precisely, it appears as the image of a morphism between two such GKZ-systems (Theorem 2.16). This result is interesting in its own, as in general there are only very few cases where geometrically interesting intersection cohomology $\mathscr{D}$-modules have an explicit description by differential operators.

Notice that the intersection cohomology $\mathscr{D}$-module mentioned above underlies a pure Hodge module. From this we can deduce a Hodge-type property of the reduced quantum $\mathscr{D}$-module (see Corollary 6.14). As already mentioned above, it cannot underly itself a Hodge module, as in general it acquires irregular singularities (this never happens for $\mathscr{D}$-modules coming from variation of Hodge structures resp. Hodge modules due to Schmid's theorem). Rather, it is part of a non-commutative Hodge (ncHodge) structure due to a key result by Sabbah ([48]).

There is another important aspect in the paper that has not yet been mentioned. The various quantum $\mathscr{D}$-modules are actually not $\mathscr{D}$-modules in the proper sense, rather, they are families of vector bundles on $\mathbb{P}^1$ together with a connection operator *with poles* along zero and infinity. This is reflected in the fact that we are looking at Fourier-Laplace transforms of certain regular $\mathscr{D}$-modules (like Gauß-Manin systems) together with a given filtration. The filtration induces a lattice structure on the FL-transformed $\mathscr{D}$-module (i.e., it yields a coherent $\mathscr{O}$-submodule generating the FL-transformed $\mathscr{D}$-module). These lattices can be reconstructed by a twisted logarithmic de Rham complex (in the sense of log geometry) of an intermediate compactification of the family of Laurent polynomials. We show in Corollary 3.20 that this twisted logarithmic de Rham complex can also be explicitly described by hypergeometric equations. Notice that for this result to hold true, we have to restrict to an open subspace of the parameter space, where certain singularities at infinity of these Laurent polynomials are allowed, but not all of them. This situation is different to the one in our earlier paper [47] where we had to exclude any singularity at infinity.

The remaining part of this introduction is a rather detailed synopsis of the content of the paper. It can be read as a warm-up, where the main playing characters are introduced together with some examples which illustrates the constructions done later.

Our main case of interest is the following: Let X_Σ be an n-dimensional smooth projective toric variety. Suppose that $\mathscr{L}_1 = \mathscr{O}_{X_\Sigma}(L_1), \ldots, \mathscr{L}_c = \mathscr{O}_{X_\Sigma}(L_c)$ are ample line bundles on X_Σ such that $-K_{X_\Sigma} - \sum_{j=1}^{c} L_j$ is nef (for many intermediate results, we can actually relax both assumptions and suppose only that the individual bundles $\mathscr{L}_1, \ldots, \mathscr{L}_c$ are nef). Put $\mathscr{E} := \bigoplus_{j=1}^{c} \mathscr{L}_j$, then $\mathscr{E}$ is a convex vector bundle. We will be interested in several quantum $\mathscr{D}$-modules, which correspond to twisted Gromov-Witten invariants of $(X_\Sigma, \mathscr{E})$ as well as to Gromov-Witten invariants on the ambient cohomology of the complete intersection $Y := s^{-1}(0)$ defined by a generic section $s \in \Gamma(X_\Sigma, \mathscr{E})$. Let us consider the total space $\mathbb{V}(\mathscr{E}^\vee)$, which is a quasi-projective toric variety with defining fan Σ'. We set $\Sigma'(1) = \{\mathbb{R}_{\geq 0}\underline{b}_1, \ldots, \mathbb{R}_{\geq 0}\underline{b}_t\}$, where the vectors $\underline{b}_i$ are the primitive integral generators of the rays of Σ'. From this set of data one can construct Lefschetz fibrations, that is, family of hyperplane sections of some projective toric varieties. The actual Landau-Ginzburg models of the above toric variety (resp. of the complete intersection Y) will be obtained by restricting