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János KOLLÁR & András NÉMETHI

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Annales Scientifiques de l'École Normale Supérieure,
45, rue d'Ulm, 75230 Paris Cedex 05, France.
Tél. : (33) 1 44 32 20 88. Fax : (33) 1 44 32 20 80.
`annales@ens.fr`

Édition / *Publication*

Société Mathématique de France
Institut Henri Poincaré
11, rue Pierre et Marie Curie
75231 Paris Cedex 05
Tél. : (33) 01 44 27 67 99
Fax : (33) 01 40 46 90 96

Abonnements / *Subscriptions*

Maison de la SMF
Case 916 - Luminy
13288 Marseille Cedex 09
Fax : (33) 04 91 41 17 51
email : `smf@smf.univ-mrs.fr`

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DURFEE'S CONJECTURE ON THE SIGNATURE OF SMOOTHINGS OF SURFACE SINGULARITIES

BY János KOLLÁR AND András NÉMETHI
WITH AN APPENDIX BY Tommaso de FERNEX

Abstract. – In 1978 Durfee conjectured various inequalities between the signature σ and the geometric genus p_g of a normal surface singularity. Since then a few counter examples have been found and positive results established in some special cases.

We prove a 'strong' Durfee-type inequality for any smoothing of a Gorenstein singularity, provided that the intersection form of the resolution is unimodular. We also prove the conjectured 'weak' inequality for all hypersurface singularities and for sufficiently large multiplicity strict complete intersections. The proofs establish general inequalities valid for any numerically Gorenstein normal surface singularity.

Résumé. – En 1978 Durfee a conjecturé plusieurs inégalités entre la signature σ et le genre géométrique p_g d'une singularité normale de surface. Depuis, quelques contre-exemples ont été trouvés et des résultats positifs établis dans des cas particuliers.

Nous montrons ici une inégalité 'forte' de type Durfee pour toute lissification d'une singularité de Gorenstein, sous la condition que la forme d'intersection de la résolution est unimodulaire. Nous prouvons aussi l'inégalité 'faible' pour toute singularité d'hypersurface et pour les intersections complètes strictes de multiplicité suffisamment grande. Les preuves établissent des inégalités générales valables pour toute singularité normale et numériquement Gorenstein de surface.

1. Introduction

Durfee's conjectures. – Let $(X,0)$ be a complex analytic normal surface singularity and $\tilde{X} \to X$ a resolution. The *geometric genus* p_g is defined as $h^1(\mathcal{O}_{\tilde{X}})$. For any one-parameter smoothing with generic (Milnor) fiber F, the rank of the second homology $H_2(F,\mathbb{Z})$ is the *Milnor number* of the smoothing μ. Furthermore, $H_2(F,\mathbb{Z})$ has a natural intersection form with Sylvester invariants (μ_+,μ_0,μ_-). Then $\mu = \mu_+ + \mu_0 + \mu_-$ and $\sigma := \mu_+ - \mu_-$ is called the *signature* of the smoothing. The Milnor number and the signature usually depend on the choice of the smoothing; but if $(X,0)$ is Gorenstein, they depend only on $(X,0)$ satisfying explicit formulas. For more details see the monographs [2, 1, 17, 20] or [16, 18, 35]. Formulas for various classes of singularities can be found in [8, 9, 10, 11, 14, 15, 12, 22].

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These local invariants should be viewed as analogs of the most important global invariants: Todd genus, Euler number and signature.

Durfee proved that $2p_g = \mu_0 + \mu_+$ [5]. Furthermore, μ_0 equals the first Betti number $b_1(L_X)$ of the link L_X of $(X,0)$.

Examples show that for a surface singularity μ_- is usually large compared to the other Sylvester invariants. Equivalently, p_g is substantially smaller than μ and σ tends to be rather negative. These observations led to the formulation of Durfee's Conjectures [5].

Strong inequality. – If $(X,0)$ is an isolated complete intersection surface singularity (ICIS) then $6p_g \leq \mu$.

Weak inequality. – If $(X,0)$ is a normal surface singularity, then for any smoothing $4p_g \leq \mu + \mu_0$. Equivalently, $\sigma \leq 0$.

Semicontinuity of σ. – If $\{(X_t,0)\}_{t\in(\mathbb{C},0)}$ is a flat family of isolated surface singularities then $\sigma(X_{t=0}) \leq \sigma(X_{t\neq 0})$.

Other invariants are provided by the combinatorics of a resolution $\pi : \tilde{X} \to X$. Let s denote the number of irreducible π-exceptional curves and K the canonical class of $\tilde{X}$. Then K^2+s is independent of the resolution and, for smoothable Gorenstein singularities,

$$\mu = 12p_g + K^2 + s - \mu_0 \quad \text{and} \quad -\sigma = 8p_g + K^2 + s; \tag{1}$$

see [5, 16, 32, 35]. Therefore, an inequality of type $\mu + \mu_0 \geq C \cdot p_g$ (for some constant C) transforms into

$$(12 - C)p_g + K^2 + s \geq 0, \quad \text{or} \quad -\sigma \geq (C-4)p_g. \tag{2}$$

In particular, one can ask for these inequalities (2) even in the non-Gorenstein case.

The resolution defines the *maximal (ideal) cycle* $Z_{\max}$, which is the divisorial part of the ideal sheaf $\pi^{-1}\mathfrak{m}_{X,0} \cdot \mathcal{O}_{\tilde{X}}$ (well defined even if this ideal sheaf is not principal).

Other invariants of $(X,0)$ are the *multiplicity,* denoted by ν, and the *embedding dimension,* denoted by e.

Known results 3. – A counterexample to the *weak inequality* was given by Wahl [35, p. 240]; it is a minimally elliptic normal surface singularity (not ICIS) with $\nu = 12$, $\mu = 3$, $\mu_0 = 0$, $p_g = 1$ and $\sigma = 1$. If one combines the results from [35, 2.2(d)] with [21] or [31], examples with arbitrary large positive σ can be constructed.

Nevertheless, both the strong and the weak inequalities hold in most examples and the intrinsic structure responsible for the positivity/negativity of the signature of a given germ has not been understood.

A counterexample to the *semicontinuity* of the signature was found in [13]: the semicontinuity already fails for some degenerations of hypersurfaces with non-degenerate Newton principal part. This excludes degeneration arguments in possible proofs of the inequalities.

The articles [14, 15] show that the *strong inequality* also fails for some non-hypersurface ICIS, and without other restrictions the best that we can expect is the weak inequality.

For hypersurfaces we have the following 'positive' results:

$8p_g < \mu$ for $(X,0)$ of multiplicity 2, Tomari [33],

$6p_g \leq \mu - 2$ for $(X, 0)$ of multiplicity 3, Ashikaga [3],
$6p_g \leq \mu - \nu + 1$ for quasi-homogeneous singularities, Xu-Yau [36],
$6p_g \leq \mu$ for suspension singularities $\{g(x, y) + z^k = 0\}$, Némethi [24, 26],
$6p_g \leq \mu$ for absolutely isolated singularities, Melle-Hernández [19].
For a short proof of $\sigma \leq 0$ in the suspension case see [25].

In this note we estimate the expression $8p_g + K^2 + s$ using properties of the dual graph of the minimal resolution. For smoothable Gorenstein singularities we obtain the following.

THEOREM 4. – *Let $(X, 0)$ be a normal Gorenstein surface singularity with embedding dimension e and geometric genus p_g. Let σ denote the signature of a smoothing. Then*

1. *If the resolution intersection form is unimodular then $-\sigma \geq 2^{4-e}(p_g + 1)$.*
2. *If $(X, 0)$ is a hypersurface singularity then $-\sigma \geq p_g + s_{\min}$, where $s_{\min}$ is the number of irreducible exceptional curves in the minimal resolution.*

The intersection form is unimodular if and only if the integral homology of the link is torsion-free [23]. Part (1) is a generalization of the following result, valid for a special family of ICIS's with unimodular lattice, namely for splice type singularities of Neumann-Wahl [30]. The *Casson Invariant Conjecture,* proved in [29, 28], states that the Casson invariant of the link is minus one-eighth the signature. As the Casson invariant is additive under splicing, and each splice component is a Brieskorn complete intersection with positive Casson invariant, the negativity of the signature follows.

We prove several inequalities that hold without the Gorenstein assumption. In fact, the strategy is to prove general inequalities using the combinatorial properties of the resolution lattice. In order to simplify the technicalities we will assume that the lattice is numerically Gorenstein. Then we apply these primary inequalities in different analytic situations.

At each step we 'lose something'. Analyzing these steps should lead to better estimates in many cases. Our aim is not to over-exploit these technicalities, but to show conceptually the general principles behind the inequalities.

It seems that $-\sigma \geq 0$ for all 'sufficiently complicated' complete intersections, but we can prove this only for *strict complete intersection* singularities, where a local ring $(\mathcal{O}_{X,0}, \mathfrak{m}_{X,0})$ is called a strict complete intersection iff the corresponding graded ring $\mathrm{Gr}_{\mathfrak{m}_{X,0}}(\mathcal{O}_{X,0})$ is a complete intersection; see [4].

PROPOSITION 5. – *Fix e and consider the set of strict ICIS of embedding dimension e. Then $-\sigma$ tends to infinity whenever the multiplicity ν tends to infinity.*

EXAMPLE 6. – [14, 15] Assume that $(X, 0)$ is a homogeneous ICIS of codimension $r = e - 2$ and multidegree $(d, \ldots, d)$. If $r = 1$ then $6p_g = \mu + 1 - \nu$. For any r the inequality $4p_g \leq \mu + 1 - \nu$ is valid. Moreover, if $r \geq 2$ is fixed, then $\frac{\mu}{p_g}$ asymptotically tends to $C_{2,r} := \frac{4(r+1)}{r+1/3}$, although $C_{2,r} \cdot p_g \leq \mu + 1$ does not hold in general. (The constant 4 is the best bound valid for any d and r.) For precise formulae see [loc.cit.].

Finally we wish to emphasize that the 'strong inequality' $6p_g \leq \mu$, conjecturally valid for all hypersurface singularities, still remains open.